Kursanalys - KTH

Formulär för kursansvarig.
Kursanalysen utförs under kursens gång.
Nomenklatur: F – föreläsning, Ö – övning, R – räknestuga, L – laboration, S – seminarium)

KURSDATA Obligatorisk del

Kursens namn

Allmän relativitetsteori

Kursnummer

SH2372

Kurspoäng och poäng fördelat på exam-former

6 hp (TEN1 6 hp)

När kursen genomfördes

Läsåret 2023/2024 (period 2)

Kursansvarig och övriga lärare

Professor Tommy Ohlsson
Linda Tenhu

Undervisningstimmar, fördelat på F, Ö, R, L, S

12 x 2h F
9 x 2h Ö

Antal registrerade studenter	29
Prestationsgrad efter 1:a examenstillfället, i %	76,3
Examinationsgrad efter 1:a examenstillfället, i %	69,0

MÅL

Ange övergripande målen för kursen

Efter fullgjord kurs skall du kunna:

- Använda differentialgeometri för att beskriva ett krökt rums egenskaper och beräkna grundläggande differentialgeometriska kvantiteter.
- Härleda och använda Einsteins fältekvationer och redogöra för energi-rörelsemängds-tensorns definition och roll i dessa, redogöra för den fysikaliska tolkningen av dess komponenter och bevisa att Newtons gravitationsteori återfås i den icke-relativistiska gränsen.
- Beräkna fysikaliska storheter för testpartiklar i en given lösning till Einsteins fältekvationer, exempelvis partikelbanor och egentider.
- Redogöra för de experiment med vilka allmän relativitetsteori har testas och jämföra med förutsägelser från Newtons gravitationsteori.
- Använda Friedmann–Lemaître–Robertson–Walker-metriken för att beskriva de olika möjligheterna för hur ett homogent universum utvecklas i tiden samt beskriva idéerna bakom kosmologisk inflation och mörk energi.

Ange hur kursen är utformad för att uppfylla målen

Kursen är utformad så att föreläsningar och övningar samt egna självstudier ska leda till att studenterna kan svara på konceptuella teorifrågor och lösa omfattande skriftliga beräkningsuppgifter och därmed uppfylla målen för kursen.

Eventuellt deltagande i länkmöte före kursstart

Synpunkter från detta

-

Kursens pedagogiska utveckling I

Beskriv de förändringar som gjorts sedan förra kursomgången. (Berätta även för studenterna vid kursstart)

Kontinuerlig examination används i kursen. Vidare är Canvas fullständigt implementerat i kursen. Quizzes (4 st.) i Canvas utgör 40 % av examinationen i kursen, medan den slutliga skrivna tentamen motsvarar 60 % av examinationen i kursen.

Kontakt med studenterna under kursens gång

Studenter i årets kurs-nämnd:	Namn	E-post (lämnas blank vid webbpublicering)

Resultat av formativ mittkursenkät

Resultat av kursmöten

Kontakt med övriga lärare under kursens gång

Kommentarer

-

Kursenkät; teknologernas synpunkter Obligatorisk del

Att komma ihåg:

1) Uppmana, mha kursnämnden, till ifyllande av kursenkät i anslutning till / just efter slutexaminationen
2) Delge kursnämnden enkäten
3) Publicera enkäten under en kortare tid

Period, då enkäten var aktiv	2024-01-12 – 2024-01-29
Frågor, som adderades till standardfrågorna	• What is your overall impression of the course? • How would you rate the difficulty of the course? • Has there been much overlap with other courses? • How were the quizzes? • How was the final written exam? • What is your opinion about the course description and the administration of the course? • What is your opinion about the course literature? • How were the exercises? (Linda Tenhu) • How were the lectures? (Tommy Ohlsson) • Please enter any further comments on the course below.
Svarsfrekvens	34 %

Förändringar sedan förra genomförandet	Kursansvarig har gett kursen för tredje gången. Quizzes i Canvas har använts som 40 % av examinationen i kursen, främst för att stimulera studenterna till kontinuerlig inlärning av materialet i kursen.
Helhetsintryck	Enligt kursenkäten svarade nästan samtliga av studenterna att de var mycket eller ganska nöjda med kursen i sin helhet. Det är min uppfattning att kontinuerlig examination i form av quizzes i Canvas kan användas även i framtiden för examinationen i kursen.
Relevanta webb-länkar	-

Kursansvarigs tolkning av enkät

Positiva synpunkter	Se bilaga.
Negativa synpunkter	Se bilaga.
Var kursen relevant i förhållande till kursmålen?	-
Syn på förkunskaperna	-
Syn på undervisningsformen	Både föreläsningarna och övningarna ansågs vara mycket bra av en majoritet av studenterna.
Syn på kurslitt/ kursmaterial	Kurslitteraturen ansågs vara bra eller mycket bra av en majoritet av studenterna.
Syn på examinationen	Quizzes ansågs vara svåra eller medelsvåra av en majoritet av studenterna. Den slutliga skriftliga tentamen ansågs vara medelsvår av en majoritet av studenterna.
Speciellt intressanta kommentarer	Se bilaga.

Synpunkter från övriga lärare efter avslutad kurs

Vad fungerade bra	-
Vad fungerade mindre bra	-

Resultat av kursnämndsmöte efter examination

Studenternas sammanfattn.	-
Förslag till förändringar	-
Länk till kursnämndsprot.	-

Kursansvarigs sammanfattande berättelse

Helhetsintryck	Jag är i stort sett mycket nöjd med utfallet av kursen. Antalet studenter var nästan dubbelt så stort jämfört med föregående år. Studenterna hade goda förkunskaper. I allmänhet, enligt kursansvarig, är antalet hp för lågt för kursen – antalet hp bör ökas. Kursen innehållet väldigt mycket material i jämförelse med andra kurser och därför bör även antalet föreläsningar ökas.
Positiva synpunkter	Se bilaga med resultat av kursenkät.
Negativa synpunkter	Se bilaga med resultat av kursenkät.
Syn på förkunskaperna	-

Syn på undervisningsformen	-
Syn på kurslitt/ kursmaterial	Läroboken fungerar bra att använda som kurslitteratur i kursen, vilken är Mike Guidry, Modern General Relativity (Cambridge University Press, 2019). Föreläsningsanteckningar har även lagts upp på Canvas regelbundet.
Syn på examinationen	Jag är på det stora hela nöjd med hur examinationen har fungerat och har inga större planer på att förändra den tills nästa kursomgång.

Kursens pedagogiska utveckling II Obligatorisk del

Hur förändringarna till denna kursomgång fungerade	Jag tycker att det har varit lätt att använda Canvas i kursen. Quizzes i Canvas fungerade bra och kommer förslagsvis att användas under nästa kursomgång också.
Förändringar som bör göras inför nästa kursomgång	Jag tycker att det finns några speciella förändringar av kursen som bör göras inför nästa kursomgång.

Övrigt

Kommentarer

Bilagor:

1. Kurs-PM: SH2372 General Relativity, 6 credits – Period 2, Fall 2023, Academic Year 2023–2024
2. Ordinarie tentamen [Final Written Exam in SH2372 General Relativity, 6 credits – January 10, 2024, 14:00–19:00]
3. Kursutvärdering

Instruktioner till kursanalysformulär
1) Kursanalysformuläret fylls i interaktivt; fälten expanderar automatiskt.
2) Fyll i fälten inom en månad efter kursens slut. (Viktigt krav från KTH!)
Skicka sedan till studierektor (som vidarebefordrar till prefekt och programansvarig).
3) Försök att ge så kompletta uppgifter som möjligt.
Tänk på att kursanalysen är ett hjälpmedel inte bara för teknologerna, utan även för Dig som lärare.
4) Med "prestationsgrad" avses antalet presterade poäng hittills på kursen
(inlämningsuppgifter, projektuppgifter, laborationer etc.) dividerat med antalet möjliga poäng för de registrerade
studenterna. Med "examinationsgrad" avses antalet studenter av de registrerade, som klarat samtliga kurskrav.
Kurssekreteraren hjälper gärna till här.
5) Kontakten med studenterna:
- Etablera kursnämnd under kursens första vecka (minst två studerande, gärna genusbalanserad).
- Lämplig bonus till kursnämndsdeltagarna är fri kurslitteratur.
- Om kursnämnd ej kan etableras, skall sektionens studienämndsordförande (SNO) kontaktas genast (se www.ths.kth.se/utbildning/utbildningsradet.html för kontaktuppgifter).
- Kursnämnden skall sammanträda under kursens gång, exempelvis i halvtid. Har mittkursutvärdering genomförts, skall den diskuteras då.
- Kursnämnden skall även ha ett möte efter det att studenterna har besvarat kursutvärderingen och kursnämndens studenter fått tillgång till resultaten. Undantaget är kurser i period fyra, där mötet bör ske
direkt efter examinatioinen är avslutad för att analysen skall vara klar innan sommaren.
- Under det avslutande kursnämndsmötet bör studenterna föra protokoll. Detta protokoll skall kursansvarig
få senast en vecka efter mötet.
- Det är kursansvarigs ansvar att kalla till kursnämndsmöten.

Slutligen, tänk på:
- det är viktigt att kursanalysen tydligt *visar utvecklingen av kursens kvalitet* från ett läsår till nästa.
- möjligheten att lägga ut kursanalysen på kurshemsidan.
- spara kursanalysen till förberedelsearbetet inför nästa kursomgång.

SH2372 General Relativity, 6 credits – Period 2, Fall 2023, Academic Year 2023–2024

Examiner and course responsible

Professor Tommy Ohlsson (tohlsson@kth.se)
Department of Physics, School of Engineering Sciences, KTH Royal Institute of Technology
Visiting address: Roslagstullsbacken 21, floor 5, room A5:1029

Teachers

- Professor Tommy Ohlsson, lectures (12 x 2h.)
- Linda Tenhu, exercises (9 x 2h.)

Literature

The course literature consists of the following books:

Guidry	Mike Guidry, *Modern General Relativity – Black Holes, Gravitational Waves, and Cosmology*, Cambridge (2019)
Blennow (MB)	Mattias Blennow, *Mathematical Methods for Physics and Engineering*, CRC Press (2018)
Blennow & Ohlsson (B&O)	Mattias Blennow and Tommy Ohlsson, *300 Problems in Special and General Relativity – With Complete Solutions*, Cambridge (2021)

Guidry will be used as the main course book. MB and B&O will be used for the exercises. Note that it is not necessary to have your own copy of MB.

Additional literature

Further recommended reading:

Carroll	Sean M. Carroll, *Spacetime and Geometry – An Introduction to General Relativity*, Cambridge (2019)
Cheng	Ta-Pei Cheng, *Relativity, Gravitation and Cosmology – A Basic Introduction*, 2nd ed., Oxford (2009)
Schutz	Bernard Schutz, *A First Course in General Relativity*, 3rd ed., Cambridge (2022)
Wald	Robert M. Wald, *General Relativity*, Chicago (1984)

Carroll, Cheng, Schutz, and Wald can be used as alternative books to Guidry or as complements.

Course contents

- Local coordinates on manifolds. Covariant and contravariant vector and tensor fields. (Pseudo-) Riemann metric.
- Covariant differentiation (Christoffel symbols, Levi-Civita connection). Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
- Basic concepts in general relativity.
- Schwarzschild space-time.
- Einstein's field equations.
- The energy-momentum tensor.
- Weak field limit.
- Experimental tests of general relativity.
- Gravitational lensing. Gravitational waves.
- Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.

About the lectures, the exercises, the quizzes, and the final exam

The material presented in the lectures is based on similar material that is covered in the books by Guidry, Carroll, Cheng, Schutz, and Wald. Lecture notes will be posted on Canvas after each lecture. Note that the material for the first four lectures is extensive, and the lecturer will not be able to present all material at these lectures, but it will be included in the lecture notes.

The exercises are based on problem solving. The teaching assistant will present the problems and their solutions to some of the listed problems during the exercises (about four problems at each exercise). All listed problems are also given as PDF files with problem statements and solutions on Canvas. For the listed problems that are not solved during the exercises, you are encouraged to solve them on your own. For some exercises, there are also listed some additional problems that are not posted as PDF files on Canvas.

The course will be examined through continuous examination. During the course, there will be four scheduled one-hour quizzes with six conceptual questions and/or smaller problems each. Each quiz can give up to 10 % of the total examination score, which means that all four quizzes can give up to 40 % of the total examination score. At the end of the course, there will be a final written exam consisting of six full computational problems (similar to the problems that are solved during the exercises). The final written exam can give up to 60 % of the total examination score. In order to pass the course (and the examination TEN1), you need to achieve at least 50 % of the total examination score. This means that the quizzes are not mandatory, i.e. you can pass the course without the quizzes. However, you cannot pass the course without the final written exam, i.e. the final written exam is a requirement. Please see *Examination* and *Grades*.

Examination

			Examination score
Q1	Quiz 1	6 conceptual questions and smaller problems	10 %
Q2	Quiz 2	6 conceptual questions and smaller problems	10 %
Q3	Quiz 3	6 conceptual questions and smaller problems	10 %
Q4	Quiz 4	6 conceptual questions and smaller problems	10 %
FE	Final written exam	6 full computational problems	60 %
			100 %

Each quiz is given at a specific occasion for one hour on Canvas and only one time during the academic year. The results of the quizzes are valid during the whole academic year. The final written exam will be given twice during the academic year.

Grades

Grade	Examination score
A	≥ 90 %
B	≥ 80 %
C	≥ 70 %
D	≥ 60 %
E	≥ 50 %
F	< 50 %

If you do not have any results from the quizzes, then the highest grade that you can obtain in the course is D. Since the highest result on the quizzes corresponds to 40 %, you cannot pass the course without taking the final written exam.

Good luck with the course!

Lecture, exercise, quiz, and final exam plan

L = lecture, E = exercise, Q = quiz, FE = final exam

L1 [Mon. 30/10, 13–15; FB51] Local coordinates on manifolds. Covariant and contravariant vector and tensor fields.
Recommended reading: Guidry Chapter 2; Carroll 1.4–1.7, 2.3–2.5, 3.2; Cheng 5.2, 13.1; Schutz Chapter 5; Wald 2.2–2.4

L2 [Wed. 1/11, 13–15; FB55] (Pseudo-) Riemann metric. Covariant differentiation (Christoffel symbols, Levi-Civita connection).
Recommended reading: Guidry Chapter 3; Carroll 2.1–2.2, 2.6–2.10, Appendix A; Cheng 13.2–13.3; Schutz 6.1–6.3; Wald 2.1, Appendix A, C.1–C.2

E1 [Thu. 2/11, 15–17; FB51] MB 1.50, 2.10, 2.20, 2.21, 2.29, 9.1, 9.4, 9.7, 9.9, 9.10 (10 problems)
Additional problems: MB 2.1, 2.12, 2.26, 9.2, 9.3, 9.5, 9.11

L3 [Mon. 6/11, 14–16; FB51] Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
Recommended reading: Guidry 7.4–7.8, 8.4, 5.6; Carroll 3.3–3.10; Cheng 5.3; Schutz 6.4–6.7, 7.4; Wald Chapter 3, C.3

E2 [Wed. 8/11, 14–16; FB55] B&O Some differential geometry … & Christoffel symbols, … 2.5, 2.9, 2.33, 2.35, 2.15, 2.39, 2.16, 2.41, 2.45, 2.50 (10 problems)
Additional problems: MB 9.14, 9.15, 9.16, 9.17, 9.18, 9.24, 9.25, 9.19, 9.21, 9.26, 9.27, 9.29, 9.34, 9.35, 9.36

Q1 [Thu. 9/11, 16–17; Canvas] Quiz 1 (based on lectures L1–L3 and exercises E1–E2)

L4 [Mon. 13/11, 13–15; FB51] Basic concepts in general relativity. Schwarzschild space-time.
Recommended reading: Guidry 7.1–7.2, 6.1–6.3, 9.1; Carroll 4.1, 4.7, 5.1–5.2; Cheng 6.1, 7.1; Schutz 7.1–7.3, 10.1; Wald 1.3–1.4, 4.1, 6.1

E3 [Wed. 15/11, 13–15; FD41] B&O Killing vector fields 2.63, 2.65, 2.69, Schwarzschild metric 2.72, 2.73, Metrics, … 2.78, 2.79, 2.80 (8 problems)

L5 [Thu. 16/11, 15–17; FB51] Schwarzschild space-time (continued).
Recommended reading: Guidry 9.3, 11.1–11.4; Carroll 5.3–5.4, 5.6–5.7, 6.1–6.3; Cheng 8.1–8.2, 14.1, 14.3, Schutz 10.2, 10.4–10.6, 11.2; Wald 6.2, 6.4, Chapter 9

L6 [Mon. 20/11, 13–15; FB51] Experimental tests of general relativity.
Recommended reading: Guidry 6.4–6.5, 9.2, 9.4–9.8; Carroll 5.5; Cheng 7.3.1, 8.3; Schutz 10.7, 11.1; Wald 6.3

E4 [Tue. 21/11, 15–17; FB51] B&O Frequency shifts 2.125, 2.126, Metrics, … 2.93, 2.97, 2.98, Kruskal–Szekeres … 2.108, Schwarzschild metric 2.76 (7 problems)

Q2 [Tue. 21/11, 17–18; Canvas] Quiz 2 (based on lectures L4–L6 and exercises E3–E4)

L7 [Thu. 23/11, 15–17; FB41] Einstein's field equations. The energy-momentum tensor.
Recommended reading: Guidry 8.5, 7.3; Carroll 4.2–4.6, 5.8; Cheng 14.2; Schutz 8.1–8.2, 10.3; Wald 4.3

L8 [Mon. 27/11, 13–15; FB55] Weak field limit.
Recommended reading: Guidry 8.1–8.3, 8.6–8.8, 22.2; Carroll 7.1–7.3; Cheng 6.2–6.3, 15.1–15.2; Schutz 8.3–8.4; Wald 4.4

E5 [Tue. 28/11, 8–10; FB55] B&O Maxwell's equations … 2.53, 2.57, 2.58, 2.59, Weak field … 2.114, 2.116, 2.118 (7 problems)

L9 [Wed. 29/11, 8–10; FB55] Gravitational lensing. Gravitational waves.
Recommended reading: Guidry 17.7, 9.9, 22.1, 22.3–22.6; Carroll 8.6, 7.4–7.7; Cheng 7.2, 7.3.2, 15.3–15.4; Schutz Chapters 9, 12; Wald 6.3, 4.4

E6 [Thu. 30/11, 15–17; FB41] B&O Gravitational lensing 2.119, Metrics, … 2.82, Gravitational waves 2.133, 2.135 (4 problems)

Q3 [Fri. 1/12, 10–11; Canvas] Quiz 3 (based on lectures L7–L9 and exercises E5–E6)

L10 [Mon. 4/12, 13–15; FB51] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.
Recommended reading: Guidry 16.1, 18; Carroll 8.1–8.3; Cheng 9.1, 9.3–9.4, 10.1; Schutz 13.1–13.2; Wald 5.1–5.2

E7 [Tue. 5/12, 8–10; FB51] B&O Metrics, … 2.81, 2.103, Frequency shifts 2.131, Cosmology … 2.146 (4 problems)
Additional problems: Guidry 19.1, 19.2, 19.6

L11 [Wed. 6/12, 8–10; FB51] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy (continued).
Recommended reading: Guidry 16.2, 17.1–17.5, 19, 17.11–17.13, 21.3; Carroll 8.4–8.5, 8.7–8.8; Cheng 10.2–10.3, 14.4, 11.1–11.5; Schutz 13.3–13.4; Wald 5.3–5.4

E8 [Thu. 7/12, 15–17; FB51] B&O Cosmology … 2.147, 2.144, 2.148, 2.149 (4 problems)
Additional problems: Guidry 21.2, 21.5

Q4 [Mon. 11/12, 13–14; Canvas] Quiz 4 (based on lectures L10–L11 and exercises E7–E8)

E9 [Tue. 12/12, 15–17; FB55] Old exams

L12 [Thu. 14/12, 15–17; FB41] Extra

FE [Wed. 10/1, 14–19; FB51, FB52] Final written exam

Department of Physics

Final Written Exam in
SH2372 General Relativity, 6 credits
January 10, 2024, 14:00–19:00

Examiner: Prof. Tommy Ohlsson
Telephone: 08-790 8261 • E-mail: tohlsson@kth.se

Allowed aids: *Useful Formulas in General Relativity*

GOOD LUCK!

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1. Consider the two-dimensional curved spacetime with the metric given by

$$ds^2 = \frac{1}{y^2}\left(dt^2 - dy^2\right)$$

in coordinates $(x^\mu) = (x^0, x^1) = (t, y)$ with $t \in \mathbb{R}$ and $y \geq 0$.

a) Find the geodesic equations and the Christoffel symbols.

b) Compute the component $R^t{}_{yty}$ of the Riemann curvature tensor.

2. A torus can be parametrized using two angles θ and φ. The metric induced by a typical embedding in $\mathbb{R}^3$ corresponds to the line element

$$ds^2 = (R + \rho \sin\varphi)^2\, d\theta^2 + \rho^2 d\varphi^2.$$

a) Find the Christoffel symbols corresponding to the Levi-Civita connection of this metric.

b) Find a Killing field for the torus and the corresponding conserved quantity along geodesics of the Levi-Civita connection.

3. The Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2$$

in $\mathbb{R}^3$ induces a non-flat Lorentzian metric on the surface

$$S = \{(t, x, y) : t^2 - x^2 - y^2 = -1\}.$$

Let φ be the polar angle in the (x, y)-plane. Compute the global time difference Δt needed for a light signal to travel from a point $\varphi_0 = 0$ to a point $\varphi = \pi/2$ on S.

4. An observer in the Schwarzschild spacetime moves with fixed radial coordinate $r = R$ and fixed angular velocity $\dot{\varphi} = \omega$ in the plane $\theta = \pi/2$. Compute a) the 4-acceleration A and b) the proper acceleration α of the observer as a function of the proper period ω.

   Hint: Christoffel symbols: $\Gamma^r_{tt} = \frac{r_*(r-r_*)}{2r^3}$, $\Gamma^r_{\varphi\varphi} = -(r - r_*)\sin^2\theta$, $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$.

5. Compute the blueshift of a light signal sent from a very distant space ship and observed at the Earth. Assume that the space ship is stationary in an approximately static spacetime.

   Useful information: The distance between the Sun and the Earth is approximately $1.5 \cdot 10^{11}$ m, the speed of light is $c \simeq 3 \cdot 10^8$ m/s, $GM_\odot \simeq 1.35 \cdot 10^{20}$ m^3/s^2 and the gravitational potential of the Earth on its surface (normalized to zero at infinity) is $-6.24 \cdot 10^7$ m^2/s^2.

6. Consider the two-dimensional de Sitter universe with the metric ($c = 1$)

$$ds^2 = dt^2 - e^{2t/R}dx^2,$$

   where $R > 0$ is a constant. Find an expression for the cosmological redshift between comoving observers at $x_0 > 0$ and $x_1 > x_0$.

Useful Formulas in General Relativity

The *covariant derivatives* of a covariant vector A_ν and a contravariant vector A^ν are given by

$$A_{\nu;\mu} \equiv \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda, \qquad A^\nu_{;\mu} \equiv \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda.$$

The *parallel transport equation* for a vector A^λ and the *geodesic equations* are given by

$$\dot{x}^\mu \nabla_\mu A^\lambda = \dot{A}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu A^\nu = 0, \qquad \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0.$$

The *torsion* T and the *curvature* R are defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

The components of the *torsion tensor* and the *Riemann curvature tensor* may be computed as

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \qquad R^\omega{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\omega_{\nu\lambda} - \partial_\nu \Gamma^\omega_{\mu\lambda} + \Gamma^\omega_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\omega_{\nu\rho}\Gamma^\rho_{\mu\lambda}.$$

By definition, X is a Killing vector field if (for all indices μ and ν)

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0.$$

The *Lie derivative* of the metric tensor $g_{\mu\nu}$ with respect to a vector field $X = X^\lambda \partial_\lambda$ is given by

$$\mathcal{L}_X g_{\mu\nu} = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu X^\lambda + g_{\mu\lambda}\partial_\nu X^\lambda.$$

The particular value $r = r_* \equiv 2GM$ represents the *Schwarzschild radius*.
The *Kruskal–Szekeres metric* is given by

$$ds^2 = \frac{16\mu^2}{r} e^{(2\mu - r)/(2\mu)} du dv - r^2 d\Omega^2, \quad uv = (2\mu - r) e^{(r-2\mu)/(2\mu)} < \frac{2GM}{e}, \quad \mu \equiv GM.$$

The *Einstein–Hilbert action* is

$$\mathscr{S}_{\rm EH} = -\frac{M_{\rm Pl}^2}{2} \int R \sqrt{|g|}\, d^4x, \quad M_{\rm Pl} \equiv \frac{1}{\sqrt{8\pi G}}.$$

Einstein's field equations in matter are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

The *energy-momentum tensor* is generally given by

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta \mathscr{S}_{\rm matter}}{\delta g^{\mu\nu}}.$$

For *an ideal (or perfect) fluid*, it holds that $T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu - p\, g_{\mu\nu}$.
Given two static observers A and B in a static spacetime, signals sent from A to B with frequencies f_A and f_B, respectively, will be redshifted according to

$$z = \frac{f_A}{f_B} - 1 = \frac{\varphi(x_B)}{\varphi(x_A)} - 1.$$

The two independent *Friedmann equations* are

$$\frac{\dot{a}(t)^2}{a(t)^2} = H(t)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a(t)^2}, \qquad \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho + 3p).$$

Solutions to Final Written Exam in General Relativity
SH2372 General Relativity • 6 ECTS credits
Wednesday January 10, 2024, 14:00 – 19:00

1. a) Using Euler–Lagrange equations for the t and y coordinates with the Lagrangian

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{y^2}\left(\dot{t}^2 - \dot{y}^2\right),$$

we find that

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{t}} = 0, \quad \frac{\partial \mathcal{L}}{\partial t} = 0, \quad \frac{\partial \mathcal{L}}{\partial \dot{t}} = \frac{2}{y^2}\dot{t} \quad \Longrightarrow \quad \frac{d}{d\tau}\frac{\dot{t}}{y^2} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = -\frac{2}{y^3}\left(\dot{t}^2 - \dot{y}^2\right), \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = -\frac{2}{y^2}\dot{y} \quad \Longrightarrow \quad -\frac{2}{y^3}\left(\dot{t}^2 - \dot{y}^2\right) + \frac{d}{d\tau}\frac{2\dot{y}}{y^2} = 0,$$

which lead to the geodesic equations

$$\ddot{t} - \frac{2}{y}\dot{t}\dot{y} = 0, \quad \ddot{y} - \frac{1}{y}\dot{t}^2 - \frac{1}{y}\dot{y}^2 = 0.$$

Thus, using the two geodesic equations for the t and y coordinates, we identify the non-zero Christoffel symbols as

$$\Gamma^t_{ty} = \Gamma^t_{yt} = \Gamma^y_{tt} = \Gamma^y_{yy} = -\frac{1}{y}.$$

b) The Christoffel symbols can be written in matrix form as

$$\Gamma^\bullet_{t\bullet} = \begin{pmatrix} \Gamma^t_{tt} & \Gamma^t_{ty} \\ \Gamma^y_{tt} & \Gamma^y_{ty} \end{pmatrix} = -\frac{1}{y}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^\bullet_{y\bullet} = \begin{pmatrix} \Gamma^t_{yt} & \Gamma^t_{yy} \\ \Gamma^y_{yt} & \Gamma^y_{yy} \end{pmatrix} = -\frac{1}{y}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, the Riemann curvature tensor in matrix form is given by

$$R^\bullet{}_{\bullet\mu\nu} = \partial_\mu \Gamma^\bullet_{\nu\bullet} - \partial_\nu \Gamma^\bullet_{\mu\bullet} + \Gamma^\bullet_{\mu\bullet}\Gamma^\bullet_{\nu\bullet} - \Gamma^\bullet_{\nu\bullet}\Gamma^\bullet_{\mu\bullet} = \partial_\mu \Gamma^\bullet_{\nu\bullet} - \partial_\nu \Gamma^\bullet_{\mu\bullet} = -R^\bullet{}_{\bullet\nu\mu}.$$

Note that $\Gamma^\bullet_{\mu\bullet}\Gamma^\bullet_{\nu\bullet} - \Gamma^\bullet_{\nu\bullet}\Gamma^\bullet_{\mu\bullet} = 0$, since $\Gamma^\bullet_{y\bullet} \propto \mathbb{1}_2$. Therefore, we have

$$R^\bullet{}_{\bullet ty} = \partial_t \Gamma^\bullet_{y\bullet} - \partial_y \Gamma^\bullet_{t\bullet} = -\partial_y \Gamma^\bullet_{t\bullet} = -\partial_y \begin{pmatrix} 0 & -\frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{y^2} \\ -\frac{1}{y^2} & 0 \end{pmatrix} = -R^\bullet{}_{\bullet yt}.$$

Note that $\partial_t \Gamma^\bullet_{y\bullet} = 0$, since $\Gamma^\bullet_{y\bullet}$ is independent of t. Thus, we obtain the non-zero components of the Riemann curvature tensor as

$$R^t{}_{yty} = R^y{}_{tty} = -\frac{1}{y^2} = -R^t{}_{yyt} = -R^y{}_{tyt}.$$

2. a) The geodesic equations are found by finding the stationary paths of

$$\mathscr{S} = \frac{1}{2}\int g_{ab}\dot{\chi}^a\dot{\chi}^b \, d\tau = \int \frac{1}{2}\left[(R + \rho\sin\varphi)^2\dot{\theta}^2 + \rho^2\dot{\varphi}^2\right] d\tau \equiv \int \mathcal{L}\, d\tau.$$

The Euler–Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -\frac{d}{d\tau}\left[(R + \rho\sin\varphi)^2\dot{\theta}\right] = -(R + \rho\sin\varphi)^2\ddot{\theta} - 2\rho\cos\varphi(R + \rho\sin\varphi)\dot{\varphi}\dot{\theta} = 0$$
$$\Longrightarrow \quad \ddot{\theta} + \frac{2\rho\cos\varphi}{R + \rho\sin\varphi}\dot{\varphi}\dot{\theta} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \rho\cos\varphi(R + \rho\sin\varphi)\dot{\theta}^2 - \rho^2\ddot{\varphi} = 0$$
$$\Longrightarrow \quad \ddot{\varphi} - \frac{\cos\varphi(R + \rho\sin\varphi)}{\rho}\dot{\theta}^2 = 0.$$

From this and the symmetry of the Levi-Civita connection, we can identify the non-zero Christoffel symbols to be

$$\Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = \frac{\rho\cos\varphi}{R+\rho\sin\varphi}, \quad \Gamma^{\varphi}_{\theta\theta} = -\frac{\cos\varphi}{\rho}(R+\rho\sin\varphi).$$

b) Since the metric does not depend explicitly on θ, the coordinate basis vector field ∂_θ is a Killing vector. The corresponding conserved quantity along a geodesic is given by

$$g(\partial_\theta, \dot\gamma) = g_{\theta a}\dot{\chi}^a = g_{\theta\theta}\dot\theta = (R+\rho\sin\varphi)^2\dot\theta = \text{const.}$$

3. Introduce the Lagrangian

$$\begin{aligned}\mathscr{L} &= g_{\mu\nu}\dot x^\mu \dot x^\nu = \dot t^2 - \dot r^2 - r^2\dot\varphi^2 = \{t^2 - x^2 - y^2 = -1\} = \frac{r^2}{r^2-1}\dot r^2 - \dot r^2 - r^2\dot\varphi^2 \\ &= \frac{1}{r^2-1}\dot r^2 - r^2\dot\varphi^2,\end{aligned}$$

where it holds that $t = \sqrt{r^2-1}$. The Euler–Lagrange variational equations, i.e., $\frac{d}{ds}\left(\frac{\partial\mathscr{L}}{\partial\dot x^\mu}\right) - \frac{\partial\mathscr{L}}{\partial x^\mu} = 0$, then become

$$\begin{aligned}\frac{d}{ds}\left(\frac{\dot r}{r^2-1}\right) + r\dot\varphi^2 + \frac{r\dot r^2}{(r^2-1)^2} &= 0 \quad \text{for } r, \\ \frac{d}{ds}(r^2\dot\varphi) &= 0 \quad \text{for } \varphi.\end{aligned}$$

Furthermore, along a light-like curve $\mathscr{L} = 0$, and thus, we have

$$\frac{1}{r^2-1}\dot r^2 - r^2\dot\varphi^2 = 0.$$

From the Euler–Lagrange variational equation for φ, we find that $r^2\dot\varphi = A = \text{const}$. Inserting this into the light-like condition, we obtain

$$\frac{1}{r^2-1}\dot r^2 - \frac{A^2}{r^2} = 0 \quad\Longrightarrow\quad \frac{dr}{ds} = \dot r = \frac{A}{r}\sqrt{r^2-1} \quad\Longrightarrow\quad \frac{d}{ds}\sqrt{r^2-1} = A,$$

so that $\sqrt{r^2-1} = As + s_0$ or $r = \sqrt{1+(As+s_0)^2}$. From $r^2\dot\varphi = A$, we find that

$$\frac{d\varphi}{ds} = \dot\varphi = \frac{A}{1+(As+s_0)^2},$$

which implies that

$$\varphi = \varphi_0 + \arctan(As+s_0) \equiv \varphi_0 + \arctan\tilde s.$$

Now, $t = \sqrt{r^2-1} = As+s_0 = \tilde s = \tan(\varphi-\varphi_0)$, i.e., $t = \tan(\varphi-\varphi_0)$, and finally, $\Delta\varphi \equiv \varphi - \varphi_0 = \frac{\pi}{2}$, which corresponds to $\Delta t = \tan\Delta\varphi \to \infty$. Thus, it takes an infinite global time difference Δt for a light signal to travel from the point $\varphi_0 = 0$ to the point $\varphi = \frac{\pi}{2}$ on the surface.

4. a) The world-line of the observer is given by

$$t = \beta\tau, \quad r = R, \quad \theta = \frac{\pi}{2}, \quad \varphi = \omega\tau,$$

where β is a constant and τ is the proper time. To find the constant β, we note that normalization of the 4-velocity $V = \beta\partial_t + \omega\partial_\varphi$ yields

$$g(V,V) = g_{tt}\beta^2 + g_{\varphi\varphi}\omega^2 = 1,$$

which implies that

$$\beta = \sqrt{\frac{1 + R^2\omega^2}{1 - r_*/R}},$$

where $r_* \equiv 2GM$ is the Schwarzschild radius. With the 4-velocity being $V = \beta\partial_t + \omega\partial_\varphi$, we generally find that

$$A = \nabla_V V = \beta^2\nabla_t\partial_t + \beta\omega(\nabla_t\partial_\varphi + \nabla_\varphi\partial_t) + \omega^2\nabla_\varphi\partial_\varphi = \left(\beta^2\Gamma^a_{tt} + 2\beta\omega\Gamma^a_{t\varphi} + \omega^2\Gamma^a_{\varphi\varphi}\right)\partial_a.$$

According to the hint, the relevant non-zero Christoffel symbols of the Schwarzschild metric are

$$\Gamma^r_{tt} = \frac{r_*(r - r_*)}{2r^3}, \quad \Gamma^r_{\varphi\varphi} = -(r - r_*)\sin^2\theta, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta.$$

However, $\theta = \pi/2$ leads to $\sin\theta = 1$ and $\cos\theta = 0$, resulting in the 4-acceleration

$$A = \left[\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right]\partial_r,$$

which is what we wanted to compute in a).

b) For the proper acceleration α, we use that $\alpha^2 \equiv -g(A, A)$, and therefore, we find that

$$\alpha^2 = \frac{R}{R - r_*}\left[\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right]^2$$

and consequently, we obtain the proper acceleration as

$$\alpha = \sqrt{\frac{R}{R - r_*}}\left|\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right|,$$

which is what we wanted to compute in b).

5. The gravitational potential of the Sun at the Earth is given by

$$\phi_\odot = -\frac{GM_\odot}{r},$$

where r is the distance between the Sun and the Earth. Thus, the gravitational potential is

$$\phi_\odot = -\frac{1.35 \cdot 10^{20}\ \mathrm{m}^3/\mathrm{s}^2}{1.5 \cdot 10^{11}\ \mathrm{m}} = -9 \cdot 10^8\ \mathrm{m}^2/\mathrm{s}^2,$$

which is an order of magnitude larger than the gravitational potential of the Earth at its surface. Thus, we can neglect the influence of the gravitational field of the Earth. The relation between the frequencies at ($\nu_\oplus$) and far away from (ν_∞) the Earth will be

$$\frac{\nu_\infty}{\nu_\oplus} \simeq \sqrt{1 + \frac{2\phi_\odot}{c^2}} \simeq 1 + \frac{\phi_\odot}{c^2}$$

for small $\phi_\odot/c^2$. The redshift parameter z is given by

$$z = \frac{\lambda_\oplus - \lambda_\infty}{\lambda_\infty} = \frac{\nu_\infty}{\nu_\oplus} - 1 \simeq \frac{\phi_\odot}{c^2} = -\frac{9 \cdot 10^8\ \mathrm{m}^2/\mathrm{s}^2}{(3 \cdot 10^8\ \mathrm{m/s})^2} = -10^{-8}$$

i.e., a blueshift, since $z < 0$. The blueshift -10^{-8} is what we wanted to compute in this problem.

6. The metric components are independent of the coordinate x and thus ∂_x is a Killing field, implying that

$$k = -g(N, \partial_x) = -g_{xx} N^x,$$

where N is the 4-frequency of the light signal, is a constant. Since the 4-frequency is a null vector, we also obtain

$$g(N,N) = g_{tt}(N^t)^2 + g_{xx}(N^x)^2 = 0 \quad \Longrightarrow \quad N^t = \sqrt{-\frac{g_{xx}(N^x)^2}{g_{tt}}} = \frac{k}{\sqrt{-g_{tt}g_{xx}}}.$$

The 4-velocity of a comoving observer is given by $V = \gamma\partial_t$ and normalization results in

$$g(V,V) = g_{tt}\gamma^2 = 1 \quad \Longrightarrow \quad \gamma = \frac{1}{\sqrt{g_{tt}}}.$$

The frequency observed by such a comoving observer is therefore

$$\omega = g(N,V) = g_{tt}\gamma N^t = g_{tt}\frac{1}{\sqrt{g_{tt}}}\frac{k}{\sqrt{-g_{tt}g_{xx}}} = \frac{k}{\sqrt{-g_{xx}}}.$$

The ratio between the observed frequency ω_1 and the emitted frequency ω_0 is therefore

$$\frac{\omega_1}{\omega_0} = \sqrt{\frac{g_{xx}(t_0)}{g_{xx}(t_1)}} = e^{(t_0-t_1)/R}.$$

Since the light signal travels along a null geodesic, its world-line satisfies

$$ds^2 = dt^2 - e^{2t/R}dx^2 = 0 \quad \Longrightarrow \quad \frac{dx}{dt} = e^{-t/R}.$$

Integrating this expression with the initial condition $x(t_0) = x_0$ results in

$$t_1 = -R\ln\left(e^{-t_0/R} - \frac{x_1 - x_0}{R}\right)$$

and inserting this into the frequency ratio leads to

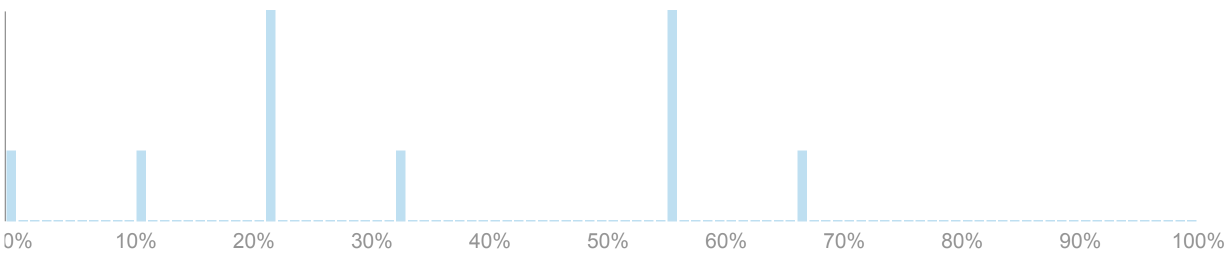
$$z = \frac{\omega_0}{\omega_1} - 1 = \frac{Re^{-t_0/R}}{Re^{-t_0/R} - x_1 + x_0} - 1 = \frac{x_1 - x_0}{Re^{-t_0/R} - x_1 + x_0},$$

which is what we wanted to compute in this problem.

Quiz Summary

Section Filter ▾ | Student analysis | Item analysis

Average score	High score	Low score	Standard deviation	Average time
0%	0%	0%	1.92	06:06



Question Breakdown

Attempts: 10 out of 10

+0.72

Discrimination Index

What is your overall impression of the course?

Very positive	7 respondents	**70 %**
Quite postive	2 respondents	20 %
Neutral - no opinion	1 respondent	10 %
Quite negative		0 %
Very negative		0 %

70%
answered
correctly

Attempts: 10 out of 10

How would you rate the difficulty of the course?

+0.17

Discrimination
Index ?

Very difficult	4 respondents	**40 %**	✓
Quite difficult	5 respondents	50 %	
Average	1 respondent	10 %	
Easy		0 %	
Very easy		0 %	

40%
answered
correctly

Attempts: 10 out of 10

Has there been much overlap with other courses?

-0

Discrimination
Index ?

Far too much overlap	**0 %**	✓

Some overlap, but it was useful to go over the topics again	7 respondents	70 %
Mostly unnecessary overlap		0 %
No overlap	3 respondents	30 %

0% answered correctly

Attempts: 10 out of 10

+0.23 Discrimination Index

How were the quizzes?

Very difficult	2 respondents	**20 %**	✓
Difficult	4 respondents	40 %	
Average	4 respondents	40 %	
Easy		0 %	
Very easy		0 %	

20% answered correctly

Attempts: 10 out of 10

+0.33 Discrimination Index

How was the final written exam?

Very difficult	1 respondent	**10 %**	✓
Difficult	1 respondent	10 %	
Average	8 respondents	80 %	
Easy		0 %	
Very easy		0 %	

10%
answered
correctly

Attempts: 9 out of 10

What is your opinion about the course description and the administration of the course?

+0.91

Discrimination Index ?

Very good	4 respondents	**40 %**	✓
Good	3 respondents	30 %	
Average	1 respondent	10 %	
Poor	1 respondent	10 %	
Very poor		0 %	
No Answer	1 respondent	10 %	

40%
answered
correctly

Attempts: 10 out of 10

What is your opinion about the course literature?

+0.42

Discrimination Index ?

Very good	3 respondents	**30 %**	✓
Good	4 respondents	40 %	
Average	2 respondents	20 %	
Poor	1 respondent	10 %	
Very poor		0 %	

30% answered correctly

Attempts: 9 out of 10

+0.51

Discrimination Index

How were the exercises? (Linda Tenhu)

Very good	5 respondents	**50 %**	✓
Good	4 respondents	40 %	
Average		0 %	
Poor		0 %	
Very poor		0 %	
No Answer	1 respondent	10 %	

50% answered correctly

Attempts: 10 out of 10

+0.78

Discrimination Index

How were the lectures? (Tommy Ohlsson)

Very good	5 respondents	**50 %**	✓
Good	4 respondents	40 %	

Average	1 respondent	10 %
Poor		0 %
Very poor		0 %

50% answered correctly

Attempts: 7 out of 10

Please enter any further comments on the course below.

Ungraded answers	10 respondents	**100 %**

The course was hard to grasp at first, which is understandable since it is not an easy topic. The course content probably correspond to 7.5 hp (would also give more time for deeper understanding), calculations were easier than the concepts. It would be nice to have the lecture notes while at the lecture, would be easier to follow the lectures. Otherwise great course.

Do not like the format with the quizzes. They should at most give bonus points to the exam. Having the quizzes count towards the total grade with 40 percent makes it practically impossible getting an A in the course. This especially since the quizzes have questions yielding a total of one point if you correctly answer six (a lot) statements, and in certain questions eight statements. And if you make a single mistake with those eight statements you get about a 1/3 deduction. I do not see how that is reasonable. Maybe single best answer would be a better format than the current one for the multiple choice questions.
Would much prefer if the grade was only given by the result of the exam with the quizzes contributing with extra points to question 1 or similar. E.g if x is the mean value of the points from the quizzes you only need to get 6-x points on question 1, if the questions gives 6 points in total. Now you are severely punished if you make a single "bad" quiz. I managed about 21/24 I think, and still had to achieve a very good score on the exam to get an A.

Some times during the lectures it was a bit difficult to follow along when the lecture notes were scrolling on the projector.

make submissions instead of quizes, feel like that would have made me understand more. The submissions could be made very hard

I really found the lecture notes to be very helpful in this course. More so than Guidry in many cases. Overall this has been one of the best courses I've taken from a structure and teaching standpoint so I think you've done a great job with it, despite the difficulty level of the material being covered.

Good, fun course!