Kursanalys - KTH

Formulär för kursansvarig.
Kursanalysen utförs under kursens gång.
Nomenklatur: F – föreläsning, Ö – övning, R – räknestuga, L – laboration, S – seminarium)

KURSDATA Obligatorisk del

Kursens namn	Kursnummer
Allmän relativitetsteori	SH2372
Kurspoäng och poäng fördelat på exam-former	**När kursen genomfördes**
6 hp (TEN1 6 hp)	Läsåret 2022/2023 (period 2)
Kursansvarig och övriga lärare	**Undervisningstimmar, fördelat på F, Ö, R, L, S**
Professor Tommy Ohlsson Linda Tenhu	12 x 2h F 9 x 2h Ö

Antal registrerade studenter	17
Prestationsgrad efter 1:a examenstillfället, i %	78,8
Examinationsgrad efter 1:a examenstillfället, i %	64,7

MÅL

Ange övergripande målen för kursen

Efter fullgjord kurs skall du kunna:

• Använda differentialgeometri för att beskriva ett krökt rums egenskaper och beräkna grundläggande differentialgeometriska kvantiteter.

• Härleda och använda Einsteins fältekvationer och redogöra för energi-rörelsemängds-tensorns definition och roll i dessa, redogöra för den fysikaliska tolkningen av dess komponenter och bevisa att Newtons gravitationsteori återfås i den icke-relativistiska gränsen.

• Beräkna fysikaliska storheter för testpartiklar i en given lösning till Einsteins fältekvationer, exempelvis partikelbanor och egentider.

• Redogöra för de experiment med vilka allmän relativitetsteori har testas och jämföra med förutsägelser från Newtons gravitationsteori.

• Använda Friedmann–Lemaître–Robertson–Walker-metriken för att beskriva de olika möjligheterna för hur ett homogent universum utvecklas i tiden samt beskriva idéerna bakom kosmologisk inflation och mörk energi.

Ange hur kursen är utformad för att uppfylla målen

Kursen är utformad så att föreläsningar och övningar samt egna självstudier ska leda till att studenterna kan svara på konceptuella teorifrågor och lösa omfattande skriftliga beräkningsuppgifter och därmed uppfylla målen för kursen.

Eventuellt deltagande i länkmöte före kursstart

Synpunkter från detta

-

Kursens pedagogiska utveckling I

Beskriv de förändringar som gjorts sedan förra kursomgången. (Berätta även för studenterna vid kursstart)

Kontinuerlig examination används i kursen. Vidare är Canvas fullständigt implementerat i kursen. Quizzes (4 st.) i Canvas utgör 40 % av examinationen i kursen, medan den slutliga skrivna tentamen motsvarar 60 % av examinationen i kursen.

Kontakt med studenterna under kursens gång

Studenter i årets kurs-nämnd:	**Namn**	**E-post** (lämnas blank vid webbpublicering)

Resultat av formativ mittkursenkät

Resultat av kursmöten

Kontakt med övriga lärare under kursens gång

Kommentarer

-

Kursenkät; teknologernas synpunkter Obligatorisk del

Att komma ihåg:
1) Uppmana, mha kursnämnden, till ifyllande av kursenkät i anslutning till / just efter slutexaminationen
2) Delge kursnämnden enkäten
3) Publicera enkäten under en kortare tid

Period, då enkäten var aktiv	2023-01-22 – 2023-02-05
Frågor, som adderades till standardfrågorna	• What is your overall impression of the course? • How would you rate the difficulty of the course? • Has there been much overlap with other courses? • How were the quizzes? • How was the final written exam? • What is your opinion about the course description and the administration of the course? • What is your opinion about the course literature? • How were the exercises? (Linda Tenhu) • How were the lectures? (Tommy Ohlsson) • Please enter any further comments on the course below.
Svarsfrekvens	35 %

Förändringar sedan förra genomförandet	Kursansvarig har gett kursen för andra gången. Quizzes i Canvas har använts som 40 % av examinationen i kursen, främst för att stimulera studenterna till kontinuerlig inlärning av materialet i kursen.
Helhetsintryck	Enligt kursenkäten svarade nästan samtliga av studenterna att de var mycket eller ganska nöjda med kursen i sin helhet. Det är min uppfattning att kontinuerlig examination i form av quizzes i Canvas kan användas även i framtiden för examinationen i kursen.
Relevanta webb-länkar	-

Kursansvarigs tolkning av enkät

Positiva synpunkter	Se bilaga.
Negativa synpunkter	Se bilaga.
Var kursen relevant i förhållande till kursmålen?	-
Syn på förkunskaperna	-
Syn på undervisningsformen	Föreläsningarna ansågs vara bra av studenterna. Övningarna ansågs vara mycket bra av en majoritet av studenterna.
Syn på kurslitt/kursmaterial	Kurslitteraturen ansågs vara medelbra av studenterna.
Syn på examinationen	Både quizzes och den slutliga skriftliga tentamen ansågs vara svåra av en majoritet av studenterna.
Speciellt intressanta kommentarer	Se bilaga.

Synpunkter från övriga lärare efter avslutad kurs

Vad fungerade bra	-
Vad fungerade mindre bra	-

Resultat av kursnämndsmöte efter examination

Studenternas sammanfattn.	-
Förslag till förändringar	-
Länk till kursnämndsprot.	-

Kursansvarigs sammanfattande berättelse

Helhetsintryck	Jag är i stort sett mycket nöjd med utfallet av kursen. Antalet studenter var något färre än jämfört med föregående år. Studenterna hade goda förkunskaper. I allmänhet, enligt kursansvarig, är antalet hp för lågt för kursen – antalet hp bör ökas. Kursen innehållet väldigt mycket material i jämförelse med andra kurser och därför bör även antalet föreläsningar ökas.
Positiva synpunkter	Se bilaga med resultat av kursenkät.
Negativa synpunkter	Se bilaga med resultat av kursenkät.
Syn på förkunskaperna	-
Syn på undervisningsformen	-

Syn på kurslitt/kursmaterial	Läroboken fungerar bra att använda som kurslitteratur i kursen, vilken är Mike Guidry, Modern General Relativity (Cambridge University Press, 2019). Föreläsningsanteckningar har även lagts upp på Canvas regelbundet.
Syn på examinationen	Jag är på det stora hela nöjd med hur examinationen har fungerat och har inga större planer på att förändra den tills nästa kursomgång.

Kursens pedagogiska utveckling II Obligatorisk del

Hur förändringarna till denna kursomgång fungerade	Jag tycker att det har varit lätt att använda Canvas i kursen. Quizzes i Canvas fungerade bra och kommer förslagsvis att användas under nästa kursomgång också.
Förändringar som bör göras inför nästa kursomgång	Jag tycker att det finns några speciella förändringar av kursen som bör göras inför nästa kursomgång.

Övrigt

Kommentarer

Bilagor:

1. Kurs-PM: SH2372 General Relativity, 6 credits – Period 2, Fall 2022, Academic Year 2022–2023
2. Ordinarie tentamen [Final Written Exam in SH2372 General Relativity, 6 credits – January 16, 2023, 14:00–19:00]
3. Kursutvärdering

Instruktioner till kursanalysformulär

1) Kursanalysformuläret fylls i interaktivt; fälten expanderar automatiskt.
2) Fyll i fälten inom en månad efter kursens slut. (Viktigt krav från KTH!)
Skicka sedan till studierektor (som vidarebefordrar till prefekt och programansvarig).
3) Försök att ge så kompletta uppgifter som möjligt.
Tänk på att kursanalysen är ett hjälpmedel inte bara för teknologerna, utan även för Dig som lärare.
4) Med "prestationsgrad" avses antalet presterade poäng hittills på kursen
(inlämningsuppgifter, projektuppgifter, laborationer etc.) dividerat med antalet möjliga poäng för de registrerade
studenterna. Med "examinationsgrad" avses antalet studenter av de registrerade, som klarat samtliga kurskrav.
Kurssekreteraren hjälper gärna till här.
5) Kontakten med studenterna:

- Etablera kursnämnd under kursens första vecka (minst två studerande, gärna genusbalanserad).
- Lämplig bonus till kursnämndsdeltagarna är fri kurslitteratur.
- Om kursnämnd ej kan etableras, skall sektionens studienämndsordförande (SNO) kontaktas genast (se www.ths.kth.se/utbildning/utbildningsradet.html för kontaktuppgifter).
- Kursnämnden skall sammanträda under kursens gång, exempelvis i halvtid. Har mittkursutvärdering genomförts, skall den diskuteras då.
- Kursnämnden skall även ha ett möte efter det att studenterna har besvarat kursutvärderingen och kursnämndens studenter fått tillgång till resultaten. Undantaget är kurser i period fyra, där mötet bör ske direkt efter examinatioinen är avslutad för att analysen skall vara klar innan sommaren.
- Under det avslutande kursnämndsmötet bör studenterna föra protokoll. Detta protokoll skall kursansvarig få senast en vecka efter mötet.
- Det är kursansvarigs ansvar att kalla till kursnämndsmöten.

Slutligen, tänk på:

- det är viktigt att kursanalysen tydligt *visar utvecklingen av kursens kvalitet* från ett läsår till nästa.
- möjligheten att lägga ut kursanalysen på kurshemsidan.
- spara kursanalysen till förberedelsearbetet inför nästa kursomgång.

SH2372 General Relativity, 6 credits – Period 2, Fall 2022, Academic Year 2022–2023

Examiner and course responsible

Professor Tommy Ohlsson (tohlsson@kth.se)
Department of Physics, School of Engineering Sciences, KTH Royal Institute of Technology
Visiting address: Roslagstullsbacken 21, floor 5, room A5:1029

Teachers

- Professor Tommy Ohlsson, lectures (12 x 2h.)
- Linda Tenhu, exercises (9 x 2h.)

Literature

The course literature consists of the following books:

Guidry	Mike Guidry, *Modern General Relativity – Black Holes, Gravitational Waves, and Cosmology*, Cambridge (2019)
Blennow (MB)	Mattias Blennow, *Mathematical Methods for Physics and Engineering*, CRC Press (2018)
Blennow & Ohlsson (B&O)	Mattias Blennow and Tommy Ohlsson, *300 Problems in Special and General Relativity – With Complete Solutions*, Cambridge (2021)

Guidry will be used as the main course book. MB and B&O will be used for the exercises. Note that it is not necessary to have your own copy of MB.

Additional literature

Further recommended reading:

Carroll	Sean M. Carroll, *Spacetime and Geometry – An Introduction to General Relativity*, Cambridge (2019)
Cheng	Ta-Pei Cheng, *Relativity, Gravitation and Cosmology – A Basic Introduction*, 2nd ed., Oxford (2009)
Schutz	Bernard Schutz, *A First Course in General Relativity*, 3rd ed., Cambridge (2022)
Wald	Robert M. Wald, *General Relativity*, Chicago (1984)

Carroll, Cheng, Schutz, and Wald can be used as alternative books to Guidry or as complements.

Course contents

- Local coordinates on manifolds. Covariant and contravariant vector and tensor fields. (Pseudo-) Riemann metric.
- Covariant differentiation (Christoffel symbols, Levi-Civita connection). Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
- Basic concepts in general relativity.
- Schwarzschild space-time.
- Einstein's field equations.
- The energy-momentum tensor.
- Weak field limit.
- Experimental tests of general relativity.
- Gravitational lensing. Gravitational waves.
- Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.

About the lectures, the exercises, the quizzes, and the final exam

The material presented in the lectures is based on similar material that is covered in the books by Guidry, Carroll, Cheng, Schutz, and Wald. Lecture notes will be posted on Canvas after each lecture. Note that the material for the first four lectures is extensive, and the lecturer will not be able to present all material at these lectures, but it will be included in the lecture notes.

The exercises are based on problem solving. The teaching assistant will present the problems and their solutions to some of the listed problems during the exercises (about four problems at each exercise). All listed problems are also given as PDF files with problem statements and solutions on Canvas. For the listed problems that are not solved during the exercises, you are encouraged to solve them on your own. For some exercises, there are also listed some additional problems that are not posted as PDF files on Canvas.

The course will be examined through continuous examination. During the course, there will be four scheduled one-hour quizzes with six conceptual questions and/or smaller problems each. Each quiz can give up to 10 % of the total examination score, which means that all four quizzes can give up to 40 % of the total examination score. At the end of the course, there will be a final written exam consisting of six full computational problems (similar to the problems that are solved during the exercises). The final written exam can give up to 60 % of the total examination score. In order to pass the course (and the examination TEN1), you need to achieve at least 50 % of the total examination score. This means that the quizzes are not mandatory, i.e. you can pass the course without the quizzes. However, you cannot pass the course without the final written exam, i.e. the final written exam is a requirement. Please see *Examination* and *Grades*.

Examination

			Examination score
Q1	Quiz 1	6 conceptual questions and smaller problems	10 %
Q2	Quiz 2	6 conceptual questions and smaller problems	10 %
Q3	Quiz 3	6 conceptual questions and smaller problems	10 %
Q4	Quiz 4	6 conceptual questions and smaller problems	10 %
FE	Final written exam	6 full computational problems	60 %
			100 %

Each quiz is given at a specific occasion for one hour on Canvas and only one time during the academic year. The results of the quizzes are valid during the whole academic year. The final written exam will be given twice during the academic year.

Grades

Grade	*Examination score*
A	≥ 90 %
B	≥ 80 %
C	≥ 70 %
D	≥ 60 %
E	≥ 50 %
F	< 50 %

If you do not have any results from the quizzes, then the highest grade that you can obtain in the course is D. Since the highest result on the quizzes corresponds to 40 %, you cannot pass the course without taking the final written exam.

Good luck with the course!

Lecture, exercise, quiz, and final exam plan

L = lecture, E = exercise, Q = quiz, FE = final exam

L1 [Mon. 31/10, 13–15; FB55] Local coordinates on manifolds. Covariant and contravariant vector and tensor fields.
Recommended reading: Guidry Chapter 2; Carroll 1.4–1.7, 2.3–2.5, 3.2; Cheng 5.2, 13.1; Schutz Chapter 5; Wald 2.2–2.4

L2 [Wed. 2/11, 10–12; FB51] (Pseudo-) Riemann metric. Covariant differentiation (Christoffel symbols, Levi-Civita connection).
Recommended reading: Guidry Chapter 3; Carroll 2.1–2.2, 2.6–2.10, Appendix A; Cheng 13.2–13.3; Schutz 6.1–6.3; Wald 2.1, Appendix A, C.1–C.2

E1 [Thu. 3/11, 15–17; FB51] MB 1.50, 2.10, 2.20, 2.21, 2.29, 9.1, 9.4, 9.7, 9.9, 9.10 (10 problems)
Additional problems: MB 2.1, 2.12, 2.26, 9.2, 9.3, 9.5, 9.11

L3 [Mon. 7/11, 14–16; FD41] Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
Recommended reading: Guidry 7.4–7.8, 8.4, 5.6; Carroll 3.3–3.10; Cheng 5.3; Schutz 6.4–6.7, 7.4; Wald Chapter 3, C.3

E2 [Wed. 9/11, 10–12; FB55] B&O Some differential geometry … & Christoffel symbols, … 2.5, 2.9, 2.33, 2.35, 2.15, 2.39, 2.16, 2.41, 2.45, 2.50 (10 problems)
Additional problems: MB 9.14, 9.15, 9.16, 9.17, 9.18, 9.24, 9.25, 9.19, 9.21, 9.26, 9.27, 9.29, 9.34, 9.35, 9.36

Q1 [Thu. 10/11, 15–16; Canvas] Quiz 1 (based on lectures L1–L3 and exercises E1–E2)

L4 [Mon. 14/11, 13–15; FD41] Basic concepts in general relativity. Schwarzschild space-time.
Recommended reading: Guidry 7.1–7.2, 6.1–6.3, 9.1; Carroll 4.1, 4.7, 5.1–5.2; Cheng 6.1, 7.1; Schutz 7.1–7.3, 10.1; Wald 1.3–1.4, 4.1, 6.1

E3 [Wed. 16/11, 10–12; FB55] B&O Killing vector fields 2.63, 2.65, 2.69, Schwarzschild metric 2.72, 2.73, Metrics, … 2.78, 2.79, 2.80 (8 problems)

L5 [Thu. 17/11, 15–17; FB51] Schwarzschild space-time (continued).
Recommended reading: Guidry 9.3, 11.1–11.4; Carroll 5.3–5.4, 5.6–5.7, 6.1–6.3; Cheng 8.1–8.2, 14.1, 14.3, Schutz 10.2, 10.4–10.6, 11.2; Wald 6.2, 6.4, Chapter 9

L6 [Mon. 21/11, 13–15; FD41] Experimental tests of general relativity.
Recommended reading: Guidry 6.4–6.5, 9.2, 9.4–9.8; Carroll 5.5; Cheng 7.3.1, 8.3; Schutz 10.7, 11.1; Wald 6.3

E4 [Wed. 23/11, 9–11; FB55] B&O Frequency shifts 2.125, 2.126, Metrics, … 2.93, 2.97, 2.98, Kruskal–Szekeres … 2.108, Schwarzschild metric 2.76 (7 problems)

Q2 [Wed. 23/11, 13–14; Canvas] Quiz 2 (based on lectures L4–L6 and exercises E3–E4)

L7 [Thu. 24/11, 15–17; FB51] Einstein's field equations. The energy-momentum tensor.
Recommended reading: Guidry 8.5, 7.3; Carroll 4.2–4.6, 5.8; Cheng 14.2; Schutz 8.1–8.2, 10.3; Wald 4.3

L8 [Mon. 28/11, 13–15; FB51] Weak field limit.
Recommended reading: Guidry 8.1–8.3, 8.6–8.8, 22.2; Carroll 7.1–7.3; Cheng 6.2–6.3, 15.1–15.2; Schutz 8.3–8.4; Wald 4.4

E5 [Wed. 30/11, 8–10; FB55] B&O Maxwell's equations … 2.53, 2.57, 2.58, 2.59, Weak field … 2.114, 2.116, 2.118 (7 problems)

L9 [Wed. 30/11, 10–12; FB51] Gravitational lensing. Gravitational waves.
Recommended reading: Guidry 17.7, 9.9, 22.1, 22.3–22.6; Carroll 8.6, 7.4–7.7; Cheng 7.2, 7.3.2, 15.3–15.4; Schutz Chapters 9, 12; Wald 6.3, 4.4

E6 [Thu. 1/12, 15–17; FB51] B&O Gravitational lensing 2.119, Metrics, … 2.82, Gravitational waves 2.133, 2.135 (4 problems)

Q3 [Thu. 2/12, 10–11; Canvas] Quiz 3 (based on lectures L7–L9 and exercises E5–E6)

L10 [Mon. 5/12, 13–15; FD41] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.
Recommended reading: Guidry 16.1, 18; Carroll 8.1–8.3; Cheng 9.1, 9.3–9.4, 10.1; Schutz 13.1–13.2; Wald 5.1–5.2

E7 [Wed. 7/12, 8–10; FB51] B&O Metrics, … 2.81, 2.103, Frequency shifts 2.131, Cosmology … 2.146 (4 problems)
Additional problems: Guidry 19.1, 19.2, 19.6

L11 [Wed. 7/12, 10–12; FB51] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy (continued).
Recommended reading: Guidry 16.2, 17.1–17.5, 19, 17.11–17.13, 21.3; Carroll 8.4–8.5, 8.7–8.8; Cheng 10.2–10.3, 14.4, 11.1–11.5; Schutz 13.3–13.4; Wald 5.3–5.4

E8 [Thu. 8/12, 15–17; FB51] B&O Cosmology … 2.147, 2.144, 2.148, 2.149 (4 problems)
Additional problems: Guidry 21.2, 21.5

Q4 [Mon. 12/12, 13–14; Canvas] Quiz 4 (based on lectures L10–L11 and exercises E7–E8)

L12 [Wed. 14/12, 10–12; FB55] Extra

E9 [Thu. 15/12, 15–17; FD41] Old exams

FE [Mon. 16/1, 14–19; FD5] Final written exam

Department of Physics

Final Written Exam in SH2372 General Relativity, 6 credits
January 16, 2023, 14:00–19:00

Examiner: Prof. Tommy Ohlsson
(Telephone: 08-790 8261 • E-mail: `tohlsson@kth.se`)

Allowed aids: *Useful Formulas in General Relativity*

GOOD LUCK!

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1. A paraboloid $z = \alpha\left(x^2 + y^2\right)$, where α is a constant, locally described by the coordinates ρ and φ such that $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, is embedded in Euclidean three-dimensional space $\mathbb{R}^3$. Compute a) the induced metric tensor, b) the Christoffel symbols, and c) the components of the Riemann curvature tensor on the paraboloid.

2. In Problem 1, find a) the flow for each of the following vector fields and determine b) whether or not they are Killing vector fields:

$$K = \partial_\rho, \quad Q = \partial_\varphi.$$

3. An observer in the Schwarzschild spacetime moves with fixed radial coordinate $r = R$ and fixed angular velocity $\dot{\varphi} = \omega$ in the plane $\theta = \pi/2$. Compute a) the 4-acceleration A and b) the proper acceleration α of the observer as a function of the proper period ω.

   Hint: Christoffel symbols: $\Gamma^r_{tt} = \frac{r_*(r-r_*)}{2r^3}$, $\quad \Gamma^r_{\varphi\varphi} = -(r-r_*)\sin^2\theta$, $\quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$.

4. Consider a satellite in circular orbit around the Earth at a distance ℓ from the surface. The line element outside of the Earth can be considered to be approximated by

$$ds^2 \simeq (1+2\Phi)dt^2 - (1+2\Phi)^{-1}dr^2 - r^2 d\Omega^2,$$

   where $\Phi \equiv -GM_\oplus/r$ is the classical gravitational potential and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

   a) What is the proper time $\Delta\tau$ required for the satellite to complete a full orbit?

   b) How does this proper time $\Delta\tau$ relate to the global time Δt required for the same orbit?

5. Consider the Robertson–Walker spacetime described by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - \mathrm{e}^{2t/t_H}\left(dr^2 + r^2 d\Omega^2\right),$$

   where $g_{\mu\nu}$ is the Robertson–Walker metric tensor, t is the universal time, $t_H \approx 14$ Gyr is the Hubble time, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

   a) Compute the path $r(t)$ of a light ray emitted at time $t = t_0$ at the origin $r = 0$.

   Hint: Assume that the light ray moves along the line $\theta = \pi/2$ and $\varphi = 0$ so that $dr/dt > 0$.

   b) Compute the proper distance $\Delta d(t, r)$ between the origin $r = 0$ and a point $r > 0$ on the line $\theta = \pi/2$ and $\varphi = 0$ at fixed time t.

6. a) The first Friedmann equation can be written as

$$\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} = \frac{8\pi G}{3}\rho,$$

which can also be written as $1 - \Omega = -\kappa/\dot{a}^2$, where $\Omega \equiv \rho/\rho_c$. Assume now that κ is small compared to the energy density ρ, which mainly consists of the cosmological constant ρ_Λ, thus leading to an inflationary universe. Show that the longer this scenario is assumed to last, the closer Ω gets to one.

b) Describe the flatness problem in words and how it is solved by inflation.

Useful Formulas in General Relativity

The *covariant derivatives* of a covariant vector A_ν and a contravariant vector A^ν are given by

$$A_{\nu;\mu} \equiv \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda, \qquad A^\nu_{;\mu} \equiv \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda.$$

The *parallel transport equation* for a vector A^λ and the *geodesic equations* are given by

$$\dot{x}^\mu \nabla_\mu A^\lambda = \dot{A}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu A^\nu = 0, \qquad \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0.$$

The *torsion* T and the *curvature* R are defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

The components of the *torsion tensor* and the *Riemann curvature tensor* may be computed as

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \qquad R^\omega{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\omega_{\nu\lambda} - \partial_\nu \Gamma^\omega_{\mu\lambda} + \Gamma^\omega_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\omega_{\nu\rho} \Gamma^\rho_{\mu\lambda}.$$

By definition, X is a Killing vector field if (for all indices μ and ν)

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0.$$

The *Lie derivative* of the metric tensor $g_{\mu\nu}$ with respect to a vector field $X = X^\lambda \partial_\lambda$ is given by

$$\mathcal{L}_X g_{\mu\nu} = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu X^\lambda + g_{\mu\lambda} \partial_\nu X^\lambda.$$

The particular value $r = r_* \equiv 2GM$ represents the *Schwarzschild radius*.
The *Kruskal–Szekeres metric* is given by

$$ds^2 = \frac{16\mu^2}{r} e^{(2\mu - r)/(2\mu)} du dv - r^2 d\Omega^2, \quad uv = (2\mu - r) e^{(r-2\mu)/(2\mu)} < \frac{2GM}{e}, \quad \mu \equiv GM.$$

The *Einstein–Hilbert action* is

$$\mathscr{S}_{\text{EH}} = -\frac{M_{\text{Pl}}^2}{2} \int R \sqrt{|g|}\, d^4x, \quad M_{\text{Pl}} \equiv \frac{1}{\sqrt{8\pi G}}.$$

Einstein's field equations in matter are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

The *energy-momentum tensor* is generally given by

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta \mathscr{S}_{\text{matter}}}{\delta g^{\mu\nu}}.$$

For *an ideal (or perfect) fluid*, it holds that $T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu - p\, g_{\mu\nu}$.
Given two static observers A and B in a static spacetime, signals sent from A to B with frequencies f_A and f_B, respectively, will be redshifted according to

$$z = \frac{f_A}{f_B} - 1 = \frac{\varphi(x_B)}{\varphi(x_A)} - 1.$$

The two independent *Friedmann equations* are

$$\frac{\dot{a}(t)^2}{a(t)^2} = H(t)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a(t)^2}, \qquad \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} (\rho + 3p)\,.$$

Solutions to Final Written Exam in General Relativity
SH2372 General Relativity • 6 ECTS credits
Monday January 16, 2023, 14:00 – 19:00

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1. a) Using the given coordinates ρ and φ such that $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$, the remaining coordinate z of the embedding on the paraboloid is given by

$$z = \alpha(x^2 + y^2) = \alpha\rho^2,$$

where α is a constant. Differentiating the coordinate functions in $\mathbb{R}^3$ leads to

$$\begin{aligned} dx &= \cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi, \\ dy &= \sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi, \\ dz &= 2\alpha\rho\, d\rho. \end{aligned}$$

Inserting this into the Euclidean metric on $\mathbb{R}^3$, we find that

$$\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 &= (\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi)^2 + (\sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi)^2 + 4\alpha^2\rho^2 d\rho^2 \\ &= (1 + 4\alpha^2\rho^2)d\rho^2 + \rho^2 d\varphi^2. \end{aligned}$$

Therefore, the non-zero components of the induced metric of the embedding on the paraboloid are

$$g_{\rho\rho} = 1 + 4\alpha^2\rho^2, \quad g_{\varphi\varphi} = \rho^2.$$

Thus, the induced metric tensor is given by

$$g = (g_{\mu\nu}) = \operatorname{diag}\left(1 + 4\alpha^2\rho^2, \rho^2\right),$$

which is what we wanted to compute in a).

b) The Christoffel symbols are now calculated directly from the induced metric to be

$$\begin{aligned} \Gamma^{\rho}_{\rho\rho} &= \frac{4\alpha^2\rho}{1 + 4\alpha^2\rho^2}, \\ \Gamma^{\rho}_{\varphi\varphi} &= -\frac{\rho}{1 + 4\alpha^2\rho^2}, \\ \Gamma^{\varphi}_{\rho\varphi} &= \Gamma^{\varphi}_{\varphi\rho} = \frac{1}{\rho}, \\ \Gamma^{\rho}_{\rho\varphi} &= \Gamma^{\rho}_{\varphi\rho} = \Gamma^{\varphi}_{\rho\rho} = \Gamma^{\varphi}_{\varphi\varphi} = 0, \end{aligned}$$

which is what we wanted to compute in b).

c) Using the Christoffel symbols, define the 2×2 matrices such that

$$\Gamma_\rho \equiv \Gamma^{\bullet}_{\rho\bullet} = \begin{pmatrix} \frac{4\alpha^2\rho}{1+4\alpha^2\rho^2} & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \quad \text{and} \quad \Gamma_\varphi \equiv \Gamma^{\bullet}_{\varphi\bullet} = \begin{pmatrix} 0 & -\frac{\rho}{1+4\alpha^2\rho^2} \\ \frac{1}{\rho} & 0 \end{pmatrix}.$$

Then, the four components of the Riemann curvature tensor, $R^{\bullet}{}_{\bullet\rho\varphi} = -R^{\bullet}{}_{\bullet\varphi\rho}$, are given by the 2×2 matrix as

$$R^{\bullet}{}_{\bullet\rho\varphi} = \partial_\rho\Gamma_\varphi - \partial_\varphi\Gamma_\rho + [\Gamma_\rho, \Gamma_\varphi] = \begin{pmatrix} 0 & \frac{4\alpha^2\rho^2}{(1+4\alpha^2\rho^2)^2} \\ -\frac{4\alpha^2}{1+4\alpha^2\rho^2} & 0 \end{pmatrix},$$

which is what we wanted to compute in c).

2. a) For $K = \partial_\rho$, the flow equations are

$$\dot{\rho} = K^\rho = 1, \quad \dot{\varphi} = K^\varphi = 0$$

with the solution $\rho = \rho_0 + \tau$, $\varphi = \varphi_0$. Similarly, for $Q = \partial_\varphi$, we find the flow equations

$$\dot{\rho} = Q^\rho = 0, \quad \dot{\varphi} = Q^\varphi = 1$$

with the solution $\rho = \rho_0$, $\varphi = \varphi_0 + \tau$, describing a rotation.

b) To determine if K or Q are Killing vector fields, we compute the Lie derivatives $\mathcal{L}_K g$ and $\mathcal{L}_Q g$, respectively. For K, we find that

$$\mathcal{L}_K g_{ab} = \partial_\rho g_{ab}.$$

In particular, we have $\mathcal{L}_K g_{\rho\rho} = 8\alpha^2\rho \neq 0$ and $\mathcal{L}_K g_{\varphi\varphi} = 2\rho \neq 0$. Therefore, the field K is not a Killing vector field. However, for Q, we obtain

$$\mathcal{L}_Q g_{ab} = \partial_\varphi g_{ab} = 0,$$

since none of the metric tensor components g_{ab} depend on φ. It follows that the field Q is a Killing vector field and the corresponding rotations are symmetries of the embedded surface.

3. a) The world line of the observer is given by

$$t = \beta\tau, \quad r = R, \quad \theta = \frac{\pi}{2}, \quad \varphi = \omega\tau,$$

where β is a constant and τ is the proper time. To find the constant β, we note that normalization of the 4-velocity $V = \beta\partial_t + \omega\partial_\varphi$ yields

$$g(V, V) = g_{tt}\beta^2 + g_{\varphi\varphi}\omega^2 = 1,$$

which implies that

$$\beta = \sqrt{\frac{1 + R^2\omega^2}{1 - r_*/R}},$$

where $r_* \equiv 2GM$ is the Schwarzschild radius. With the 4-velocity being $V = \beta\partial_t + \omega\partial_\varphi$, we generally find that

$$A = \nabla_V V = \beta^2\nabla_t\partial_t + \beta\omega(\nabla_t\partial_\varphi + \nabla_\varphi\partial_t) + \omega^2\nabla_\varphi\partial_\varphi = \left(\beta^2\Gamma^a_{tt} + 2\beta\omega\Gamma^a_{t\varphi} + \omega^2\Gamma^a_{\varphi\varphi}\right)\partial_a.$$

According to the hint, the relevant non-zero Christoffel symbols of the Schwarzschild metric are

$$\Gamma^r_{tt} = \frac{r_*(r - r_*)}{2r^3}, \quad \Gamma^r_{\varphi\varphi} = -(r - r_*)\sin^2\theta, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta.$$

However, $\theta = \pi/2$ leads to $\sin\theta = 1$ and $\cos\theta = 0$, resulting in the 4-acceleration

$$A = \left[\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right]\partial_r,$$

which is what we wanted to compute in a).

b) For the proper acceleration α, we use that $\alpha^2 \equiv -g(A, A)$, and therefore, we find that

$$\alpha^2 = \frac{R}{R - r_*}\left[\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right]^2$$

and consequently, we obtain the proper acceleration as

$$\alpha = \sqrt{\frac{R}{R - r_*}}\left|\frac{r_*}{2}\left(\frac{1}{R^2} + 3\omega^2\right) - R\omega^2\right|,$$

which is what we wanted to compute in b).

4. a) The line element outside of the Earth is given by

$$ds^2 \simeq (1+2\Phi)dt^2 - (1+2\Phi)^{-1}dr^2 - r^2 d\Omega^2,$$

where $\Phi \equiv -GM_\oplus/r$ is the gravitational potential and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. A satellite is orbiting at a distance ℓ from the surface of the Earth. Therefore, we have $r = R \equiv R_\oplus + \ell$. We are interested the proper time $\Delta\tau$ for the satellite to complete a full orbit around the Earth. Due to the spherical symmetry, we can assume $\theta = \pi/2$. Thus, the line element can be written as

$$ds^2 = (1+2\Phi)dt^2 - (1+2\Phi)^{-1}dr^2 - r^2 d\varphi^2.$$

The Lagrangian is given by

$$\mathcal{L} = (1+2\Phi)\dot{t}^2 - (1+2\Phi)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = 1.$$

The Euler–Lagrange equations are then given by

$$\ddot{r} - \frac{\Phi'}{1+2\Phi}\dot{r}^2 + \Phi'(1+2\Phi)\dot{t}^2 - r(1+2\Phi)\dot{\varphi}^2 = 0, \quad \ddot{t} = 0, \quad \ddot{\varphi} = 0,$$

where $\Phi' = \frac{d\Phi}{dr} = -\Phi/r = GM_\oplus/r^2$. The last two equations can be integrated at once to give

$$\dot{t} = \alpha, \quad \dot{\varphi} = \beta,$$

where α and β are constants. From the first equation, using $\dot{r} = 0$ and $\ddot{r} = 0$, we can determine a relation between $\dot{t}$ and $\dot{\varphi}$. Inserting the expressions for $\dot{t}$ and $\dot{\varphi}$, we obtain

$$-\frac{\Phi}{R}\alpha^2 - R\beta^2 = 0.$$

Furthermore, using $\dot{r} = 0$, the Lagrangian can be reduced to

$$\mathcal{L} = (1+2\Phi)\alpha^2 - R^2\beta^2 = 1,$$

which can be solved for $\alpha = \frac{1}{\sqrt{1+3\Phi}}$ and $\beta = \alpha\frac{\sqrt{-\Phi}}{R} = \frac{1}{R}\sqrt{\frac{-\Phi}{1+3\Phi}}$. Thus, using $\dot{\varphi} = \frac{d\varphi}{d\tau} = \beta$, we obtain

$$2\pi = \beta\Delta\tau \quad \Longleftrightarrow \quad \Delta\tau = \frac{2\pi}{\beta} = 2\pi R\sqrt{\frac{1+3\Phi}{-\Phi}} = 2\pi R\sqrt{\frac{R}{GM_\oplus} - 3},$$

which is the proper time required for the satellite to complete a full orbit.

b) Now, using $\dot{t} = \frac{dt}{d\tau} = \alpha$, we can also determine Δt in terms of $\Delta\tau$ as

$$\Delta t = \alpha\Delta\tau = \frac{\Delta\tau}{\sqrt{1+3\Phi}} = 2\pi\frac{R}{\sqrt{-\Phi}} = 2\pi\sqrt{\frac{R^3}{GM_\oplus}},$$

which is the global time required for the same orbit, and in fact, Kepler's third law.

5. a) The geodesic equation can be derived from the Lagrangian $\dot{t}^2 - \exp(2t/t_H)\dot{r}^2 = 0$ (assuming $\dot{\theta} = 0$ and $\dot{\varphi} = 0$, since $\theta = \pi/2$ and $\varphi = 0$), where $\dot{t} = dt/d\tau$, i.e., $dr/dt = \pm\exp(-t/t_H)$ with $r(t_0) = 0$. Using $dr/dt = +\exp(-t/t_H) > 0$ (according to the hint), this implies that

$$r(t) = \int_{r'=0}^{r'=r} dr' = \int_{t'=t_0}^{t'=t} \mathrm{e}^{-t'/t_H}\, dt' = \left[-t_H \mathrm{e}^{-t'/t_H}\right]_{t_0}^{t} = -t_H\left(\mathrm{e}^{-t/t_H} - \mathrm{e}^{-t_0/t_H}\right),$$

which is the path that we wanted to compute in a).

b) For the line $\theta = \pi/2$ and $\varphi = 0$ at fixed universal time $t = t_0 =$ const., we have $dt = 0$ and $d\Omega = 0$, which imply that

$$ds^2 = -\mathrm{e}^{2t/t_H}dr^2 \quad \Longrightarrow \quad g_{rr}(t) = -\mathrm{e}^{2t/t_H}.$$

Thus, the proper distance $\Delta d(t,r)$ between the origin $r = 0$ and a point $r > 0$ is given by

$$\Delta d(t,r) = \int_{r'=0}^{r'=r}\sqrt{-g_{rr}(t)}\, dr' = \int_0^r \sqrt{\mathrm{e}^{2t/t_H}}\, dr' = \mathrm{e}^{t/t_H}\int_0^r dr' = r\mathrm{e}^{t/t_H},$$

which what we wanted to compute in b).

6. a) If we use the given approximation, i.e., κ is small and $\rho \simeq \rho_\Lambda$, we can approximate the first Friedmann equation as

$$\frac{\dot{a}^2}{a^2} \simeq \frac{8\pi G}{3}\rho_\Lambda.$$

This equation has the solution

$$a(t) \simeq a_0 \mathrm{e}^{(t-t_0)/\Delta\tau}, \quad \Delta\tau \equiv \sqrt{\frac{3}{8\pi G \rho_\Lambda}}.$$

Inserting this solution into the expression for Ω, i.e., $\dot{a}(t) \simeq \frac{a_0}{\Delta\tau}\mathrm{e}^{(t-t_0)/\Delta\tau}$, we obtain

$$1 - \Omega = -\kappa \dot{a}(t)^{-2} \propto \mathrm{e}^{-2(t-t_0)/\Delta\tau}.$$

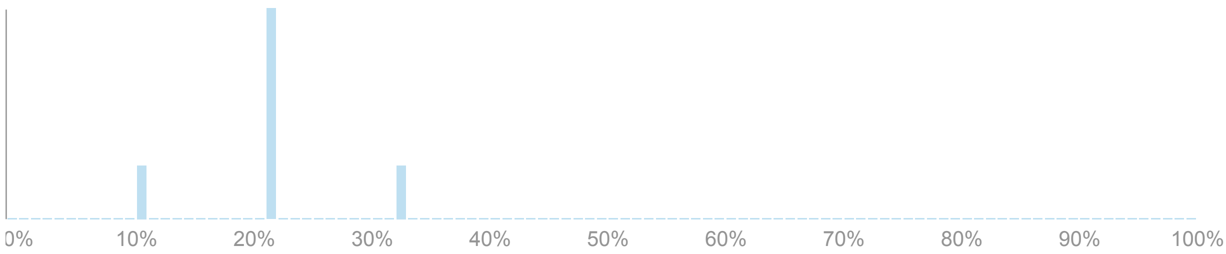
We observe that the larger t becomes, the smaller $1-\Omega$ becomes, and thus, $1-\Omega$ gets closer and closer to zero, which means that Ω gets closer and closer to one.

b) Today, we have a universe with a curvature that is very close to zero, which without inflation would need a lot of fine-tuning to maintain. After inflation, a spacetime is obtained, which is very close to being flat as per the solution to a), even if the initial spacetime had a lot of curvature.

Quiz Summary

Section Filter ▾ | Student analysis | Item analysis

Average score	High score	Low score	Standard deviation	Average time
0%	0%	0%	0.58	10:50:65



Question Breakdown

Attempts: 6 out of 6

-0.61

Discrimination Index

What is your overall impression of the course?

Very positive	2 respondents	**33 %**	✓
Quite postive	3 respondents	50 %	
Neutral - no opinion	1 respondent	17 %	
Quite negative		0 %	
Very negative		0 %	

33% answered correctly

Attempts: 6 out of 6

How would you rate the difficulty of the course?

+0.61

Discrimination Index

Very difficult	2 respondents	**33 %**	✓
Quite difficult	3 respondents	50 %	
Average	1 respondent	17 %	
Easy		0 %	
Very easy		0 %	

33% answered correctly

Attempts: 6 out of 6

Has there been much overlap with other courses?

-0

Discrimination Index ?

Far too much overlap		**0 %**	✓
Some overlap, but it was useful to go over the topics again	3 respondents	50 %	
Mostly unnecessary overlap		0 %	
No overlap	3 respondents	50 %	

0% answered correctly

Attempts: 6 out of 6

-0

How were the quizzes?

Discrimination Index ?

Very difficult		**0 %**	✓
Difficult	3 respondents	50 %	
Average	3 respondents	50 %	
Easy		0 %	
Very easy		0 %	

0% answered correctly

Attempts: 5 out of 6

How was the final written exam?

+1

Discriminination Index ?

Very difficult	1 respondent	**17 %**	✓
Difficult	2 respondents	33 %	
Average	1 respondent	17 %	
Easy	1 respondent	17 %	
Very easy		0 %	
No Answer	1 respondent	17 %	

17% answered correctly

Attempts: 5 out of 6

What is your opinion about the course description and the administration of the course?

-0.25

Discrimination Index ?

Very good	1 respondent	**17 %**	✓
Good	3 respondents	50 %	
Average		0 %	
Poor	1 respondent	17 %	
Very poor		0 %	
No Answer	1 respondent	17 %	

17% answered correctly

Attempts: 5 out of 6

What is your opinion about the course literature?

-0.25

Discrimination Index ?

Very good	1 respondent	**17 %**	✓
Good	1 respondent	17 %	
Average	2 respondents	33 %	
Poor	1 respondent	17 %	
Very poor		0 %	
No Answer	1 respondent	17 %	

17% answered correctly

Attempts: 5 out of 6

+0.25

Discrimination Index ⓘ

How were the exercises? (Linda Tenhu)

Very good	4 respondents	**67 %**	✓
Good	1 respondent	17 %	
Average		0 %	
Poor		0 %	
Very poor		0 %	
No Answer	1 respondent	17 %	

67% answered correctly

Attempts: 5 out of 6

-0.25

Discrimination Index ⓘ

How were the lectures? (Tommy Ohlsson)

Very good	1 respondent	**17 %**	✓
Good	2 respondents	33 %	
Average	2 respondents	33 %	
Poor		0 %	
Very poor		0 %	
No Answer	1 respondent	17 %	

17% answered correctly

Attempts: 4 out of 6

Please enter any further comments on the course below.

Ungraded answers 6 respondents **100 %** ✓

It was a super interesting course! One of my favorite courses I have taken, and I think it was presented well for the most part. The only things I would have liked to improve is to make the course a few more credits, spent a little more time on the mathematics and perhaps alter the grading system a little bit.
The lectures and exercises were good, ideally of course it would be given by people who do GR in their day to day but you could tell that you both found the subject interesting which is the most important in my opinion. The grading system did not leave much room for error which you can have different opinions about, but for many people it's hard to keep a steady pace which is punished in this system compared to ones where for example quizzes are used to raise grades on the final exam. But I also understand that there is so much material in so few credits so it's hard to test knowledge of all areas and still give several opportunities to be improve.
Thanks!

—

The math behind the course is hard (proving the theorems was not trivial at all). However, the final exam was more like "problem-solving", which is quite easier. Maybe you can do something about this leap in difficulty (lectures were way harder than the examinations).
Linda is a good teaching assistant, her classes were useful for the final exam. The professor knows a lot about the topic, which was great.
I liked the course. I would improve the formula sheet (by adding formulas seen in classes that, currently, are not there. That would help a lot).

—

Old exams or exemple exams would have made studying for the last exam less stressful. The exercises varied in difficulty a lot so it was difficult to know how difficult the exam was going to be.
The way the quizes worked made it very difficult to aim for higher grades since most People were not eligible for the best grader before the exam.
Very interesting content with good lectures and excellent exercises

—

I personally feel as though canvas quizes are better suited for choices than for inputs. Hence the questions where you should input your answer in text should be avoided.