Kursanalys - KTH

Formulär för kursansvarig.
Kursanalysen utförs under kursens gång.
Nomenklatur: F – föreläsning, Ö – övning, R – räknestuga, L – laboration, S – seminarium)

KURSDATA Obligatorisk del

Kursens namn	**Kursnummer**
Allmän relativitetsteori	SH2372
Kurspoäng och poäng fördelat på exam-former	**När kursen genomfördes**
6 hp (TEN1 6 hp)	Läsåret 2021/2022 (period 2)
Kursansvarig och övriga lärare	**Undervisningstimmar, fördelat på F, Ö, R, L, S**
Professor Tommy Ohlsson Linda Tenhu	12 x 2h F 9 x 2h Ö

Antal registrerade studenter	27
Prestationsgrad efter 1:a examenstillfället, i %	88,9
Examinationsgrad efter 1:a examenstillfället, i %	77,8

MÅL

Ange övergripande målen för kursen

Efter fullgjord kurs skall du kunna:

- Använda differentialgeometri för att beskriva ett krökt rums egenskaper och beräkna grundläggande differentialgeometriska kvantiteter.
- Härleda och använda Einsteins fältekvationer och redogöra för energi-rörelsemängds-tensorns definition och roll i dessa, redogöra för den fysikaliska tolkningen av dess komponenter och bevisa att Newtons gravitationsteori återfås i den icke-relativistiska gränsen.
- Beräkna fysikaliska storheter för testpartiklar i en given lösning till Einsteins fältekvationer, exempelvis partikelbanor och egentider.
- Redogöra för de experiment med vilka allmän relativitetsteori har testas och jämföra med förutsägelser från Newtons gravitationsteori.
- Använda Friedmann–Lemaître–Robertson–Walker-metriken för att beskriva de olika möjligheterna för hur ett homogent universum utvecklas i tiden samt beskriva idéerna bakom kosmologisk inflation och mörk energi.

Ange hur kursen är utformad för att uppfylla målen

Kursen är utformad så att föreläsningar och övningar samt egna självstudier ska leda till att studenterna kan svara på konceptuella teorifrågor och lösa omfattande skriftliga beräkningsuppgifter och därmed uppfylla målen för kursen.

Eventuellt deltagande i länkmöte före kursstart

Synpunkter från detta

-

Kursens pedagogiska utveckling I

Beskriv de förändringar som gjorts sedan förra kursomgången. (Berätta även för studenterna vid kursstart)

Kontinuerlig examination har införts i kursen. Vidare har Canvas implementerats fullständigt i kursen. Quizzes (4 st.) i Canvas har införts som 40 % av examinationen i kursen, medan den slutliga skrivna tentamen motsvarar 60 % av examinationen i kursen.

Kontakt med studenterna under kursens gång

Studenter i årets kurs-nämnd:	Namn	E-post (lämnas blank vid webbpublicering)

Resultat av formativ mittkursenkät

Resultat av kursmöten

Kontakt med övriga lärare under kursens gång

Kommentarer

-

Kursenkät; teknologernas synpunkter Obligatorisk del

Att komma ihåg:
1) Uppmana, mha kursnämnden, till ifyllande av kursenkät i anslutning till / just efter slutexaminationen
2) Delge kursnämnden enkäten
3) Publicera enkäten under en kortare tid

Period, då enkäten var aktiv	2022-01-16 – 2022-01-28
Frågor, som adderades till standardfrågorna	• What is your overall impression of the course? • How would you rate the difficulty of the course? • Has there been much overlap with other courses? • How were the quizzes? • How was the final written exam? • What is your opinion about the course description and the administration of the course? • What is your opinion about the course literature? • How were the exercises? (Linda Tenhu) • How were the lectures? (Tommy Ohlsson) • Please enter any further comments on the course below.
Svarsfrekvens	53 %

Förändringar sedan förra genomförandet	Kursen har en ny kursansvarig och examinator. Quizzes i Canvas har införts som 40 % av examinationen i kursen, främst för att stimulera studenterna till kontinuerlig inlärning av materialet i kursen.
Helhetsintryck	Enligt kursenkäten svarade nästan samtliga av studenterna att de var mycket eller ganska nöjda med kursen i sin helhet. Det är min uppfattning att kontinuerlig examination i form av quizzes i Canvas kan användas även i framtiden för examinationen i kursen.
Relevanta webb-länkar	-

Kursansvarigs tolkning av enkät

Positiva synpunkter	Se bilaga.
Negativa synpunkter	Se bilaga.
Var kursen relevant i förhållande till kursmålen?	-
Syn på förkunskaperna	-
Syn på undervisningsformen	Föreläsningarna ansågs vara mycket bra eller bra av en majoritet av studenterna. Även övningarna ansågs vara mycket bra eller bra av en majoritet av studenterna.
Syn på kurslitt/ kursmaterial	Kurslitteraturen ansågs vara bra eller medelbra av en majortitet av studenterna.
Syn på examinationen	Både quizzes och den slutliga skriftliga tentamen ansågs vara svåra av en majoritet av studenterna.
Speciellt intressanta kommentarer	Se bilaga.

Synpunkter från övriga lärare efter avslutad kurs

Vad fungerade bra	-
Vad fungerade mindre bra	-

Resultat av kursnämndsmöte efter examination

Studenternas sammanfattn.	-
Förslag till förändringar	-
Länk till kursnämndsprot.	-

Kursansvarigs sammanfattande berättelse

Helhetsintryck	Jag är i stort sett mycket nöjd med utfallet av kursen. Antalet studenter var något fler än jämfört med föregående år. Studenterna hade goda förkunskaper. Vissa studenter tyckte att den första delen av kursen innehöll för mycket material, vilket kursansvarig kan hålla med om. I allmänhet, enligt kursansvarig, är antalet hp för lågt för kursen – antalet hp bör ökas. Kursen innehållet väldigt mycket material i jämförelse med andra kurser och därför bör även antalet föreläsningar ökas.
Positiva synpunkter	Se bilaga med resultat av kursenkät.
Negativa synpunkter	Se bilaga med resultat av kursenkät.

Syn på förkunskaperna	-
Syn på undervisningsformen	-
Syn på kurslitt/ kursmaterial	Läroboken fungerar bra att använda som kurslitteratur i kursen, vilken är Mike Guidry, Modern General Relativity (Cambridge University Press, 2019). Föreläsningsanteckningar har även lagts upp på Canvas regelbundet.
Syn på examinationen	Jag är på det stora hela nöjd med hur examinationen har fungerat och har inga större planer på att förändra den tills nästa kursomgång.

Kursens pedagogiska utveckling II Obligatorisk del

Hur förändringarna till denna kursomgång fungerade	Jag tycker att det har varit lätt att använda Canvas i kursen. Quizzes i Canvas fungerade bra och kommer förslagsvis att användas under nästa kursomgång också.
Förändringar som bör göras inför nästa kursomgång	Jag tycker att det finns några speciella förändringar av kursen som bör göras inför nästa kursomgång.

Övrigt

Kommentarer

Bilagor:

1. Kurs-PM: SH2372 General Relativity, 6 credits – Period 2, Fall 2021, Academic Year 2021–2022
2. Ordinarie tentamen [Final Written Exam in SH2372 General Relativity, 6 credits – January 17, 2022, 08:00–13:00]
3. Resultat av: General Relativity, SH2372, ht 2021

Instruktioner till kursanalysformulär

1) Kursanalysformuläret fylls i interaktivt; fälten expanderar automatiskt.
2) Fyll i fälten inom en månad efter kursens slut. (Viktigt krav från KTH!)
   Skicka sedan till studierektor (som vidarebefordrar till prefekt och programansvarig).
3) Försök att ge så kompletta uppgifter som möjligt.
   Tänk på att kursanalysen är ett hjälpmedel inte bara för teknologerna, utan även för Dig som lärare.
4) Med ”prestationsgrad” avses antalet presterade poäng hittills på kursen (inlämningsuppgifter, projektuppgifter, laborationer etc.) dividerat med antalet möjliga poäng för de registrerade studenterna. Med ”examinationsgrad” avses antalet studenter av de registrerade, som klarat samtliga kurskrav.
   Kurssekreteraren hjälper gärna till här.
5) Kontakten med studenterna:

- Etablera kursnämnd under kursens första vecka (minst två studerande, gärna genusbalanserad).
- Lämplig bonus till kursnämndsdeltagarna är fri kurslitteratur.
- Om kursnämnd ej kan etableras, skall sektionens studienämndsordförande (SNO) kontaktas genast (se www.ths.kth.se/utbildning/utbildningsradet.html för kontaktuppgifter).
- Kursnämnden skall sammanträda under kursens gång, exempelvis i halvtid. Har mittkursutvärdering genomförts, skall den diskuteras då.
- Kursnämnden skall även ha ett möte efter det att studenterna har besvarat kursutvärderingen och kursnämndens studenter fått tillgång till resultaten. Undantaget är kurser i period fyra, där mötet bör ske direkt efter examinatioinen är avslutad för att analysen skall vara klar innan sommaren.
- Under det avslutande kursnämndsmötet bör studenterna föra protokoll. Detta protokoll skall kursansvarig få senast en vecka efter mötet.
- Det är kursansvarigs ansvar att kalla till kursnämndsmöten.

Slutligen, tänk på:

- det är viktigt att kursanalysen tydligt *visar utvecklingen av kursens kvalitet* från ett läsår till nästa.
- möjligheten att lägga ut kursanalysen på kurshemsidan.
- spara kursanalysen till förberedelsearbetet inför nästa kursomgång.

SH2372 General Relativity, 6 credits – Period 2, Fall 2021, Academic Year 2021–2022

Examiner and course responsible

Professor Tommy Ohlsson (tohlsson@kth.se)
Department of Physics, School of Engineering Sciences, KTH Royal Institute of Technology
Visiting address: Roslagstullsbacken 21, floor 5, room A5:1029

Teachers

- Professor Tommy Ohlsson, lectures (12 x 2h.)
- Linda Tenhu, exercises (9 x 2h.)

Literature

The course literature consists of the following books:

Guidry	Mike Guidry, *Modern General Relativity – Black Holes, Gravitational Waves, and Cosmology*, Cambridge (2019)
Blennow (MB)	Mattias Blennow, *Mathematical Methods for Physics and Engineering*, CRC Press (2018)
Blennow & Ohlsson (B&O)	Mattias Blennow and Tommy Ohlsson, *300 Problems in Special and General Relativity – With Complete Solutions*, Cambridge (2021)

Guidry will be used as the main course book. MB and B&O will be used for the exercises. Note that it is not necessary to have your own copy of MB.

Additional literature

Further recommended reading:

Carroll	Sean M. Carroll, *Spacetime and Geometry – An Introduction to General Relativity*, Pearson (2004)
Cheng	Ta-Pei Cheng, *Relativity, Gravitation and Cosmology – A Basic Introduction*, 2nd ed., Oxford (2009)
Schutz	Bernard Schutz, *A First Course in General Relativity*, 2nd ed., Cambridge (2009)
Wald	Robert M. Wald, *General Relativity*, Chicago (1984)

Carroll, Cheng, Schutz, and Wald can be used as alternative books to Guidry or as complements.

Course contents

- Local coordinates on manifolds. Covariant and contravariant vector and tensor fields. (Pseudo-) Riemann metric.
- Covariant differentiation (Christoffel symbols, Levi-Civita connection). Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
- Basic concepts in general relativity.
- Schwarzschild space-time.
- Einstein's field equations.
- The energy-momentum tensor.
- Weak field limit.
- Experimental tests of general relativity.
- Gravitational lensing. Gravitational waves.
- Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.

About the lectures, the exercises, the quizzes, and the final exam

The material presented in the lectures is based on similar material that is covered in the books by Guidry, Carroll, Cheng, Schutz, and Wald. Lecture notes will be posted on Canvas after each lecture. Note that the material for the first four lectures is extensive, and the lecturer will not be able to present all material at these lectures, but it will be included in the lecture notes.

The exercises are based on problem solving. The teaching assistant will present the problems and their solutions to some of the listed problems during the exercises (about four problems at each exercise). All listed problems are also given as PDF files with problem statements and solutions on Canvas. For the listed problems that are not solved during the exercises, you are encouraged to solve them on your own. For some exercises, there are also listed some additional problems that are not posted as PDF files on Canvas.

The course will be examined through continuous examination. During the course, there will be four scheduled one-hour quizzes with six conceptual questions and/or smaller problems each. Each quiz can give up to 10 % of the total examination score, which means that all four quizzes can give up to 40 % of the total examination score. At the end of the course, there will be a final written exam consisting of six full computational problems (similar to the problems that are solved during the exercises). The final written exam can give up to 60 % of the total examination score. In order to pass the course (and the examination TEN1), you need to achieve at least 50 % of the total examination score. This means that the quizzes are not mandatory, i.e. you can pass the course without the quizzes. However, you cannot pass the course without the final written exam, i.e. the final written exam is a requirement. Please see *Examination* and *Grades*.

Examination

			Examination score
Q1	Quiz 1	6 conceptual questions and smaller problems	10 %
Q2	Quiz 2	6 conceptual questions and smaller problems	10 %
Q3	Quiz 3	6 conceptual questions and smaller problems	10 %
Q4	Quiz 4	6 conceptual questions and smaller problems	10 %
FE	Final written exam	6 full computational problems	60 %
			100 %

Each quiz is given at a specific occasion for one hour on Canvas and only one time during the academic year. The results of the quizzes are valid during the whole academic year. The final written exam will be given twice during the academic year.

Grades

Grade	*Examination score*
A	≥ 90 %
B	≥ 80 %
C	≥ 70 %
D	≥ 60 %
E	≥ 50 %
F	< 50 %

If you do not have any results from the quizzes, then the highest grade that you can obtain in the course is D. Since the highest result on the quizzes corresponds to 40 %, you cannot pass the course without taking the final written exam.

Good luck with the course!

Lecture, exercise, quiz, and final exam plan

L = lecture, E = exercise, Q = quiz, FE = final exam

L1 [Mon. 1/11, 13-15] Local coordinates on manifolds. Covariant and contravariant vector and tensor fields.
Recommended reading: Guidry Chapter 2; Carroll 1.4–1.7, 2.3–2.5, 3.2; Cheng 5.2, 13.1; Schutz Chapter 5; Wald 2.2–2.4

L2 [Wed. 3/11, 10-12] (Pseudo-) Riemann metric. Covariant differentiation (Christoffel symbols, Levi-Civita connection).
Recommended reading: Guidry Chapter 3; Carroll 2.1–2.2, 2.6–2.10, Appendix A; Cheng 13.2–13.3; Schutz 6.1–6.3; Wald 2.1, Appendix A, C.1–C.2

E1 [Mon. 8/11, 13-15] MB 1.50, 2.10, 2.20, 2.21, 2.29, 9.1, 9.4, 9.7, 9.9, 9.10 (10 problems)
Additional problems: MB 2.1, 2.12, 2.26, 9.2, 9.3, 9.5, 9.11

L3 [Wed. 10/11, 10-12] Parallel transport. Curved spaces. Lie derivatives and Killing vector fields.
Recommended reading: Guidry 7.4–7.8, 8.4, 5.6; Carroll 3.3–3.10; Cheng 5.3; Schutz 6.4–6.7, 7.4; Wald Chapter 3, C.3

E2 [Mon. 15/11, 13-15] B&O Some differential geometry … & Christoffel symbols, … 2.5 [2.4], 2.9 [2.8+2.9], 2.33 [2.31+2.32+2.33], 2.35 [2.35], 2.15 [2.14], 2.39 [2.39+2.40], 2.16 [2.15], 2.41 [2.43], 2.45 [2.48], 2.50 [2.53] (10 problems)
Additional problems: MB 9.14, 9.15, 9.16, 9.17, 9.18, 9.24, 9.25, 9.19, 9.21, 9.26, 9.27, 9.29, 9.34, 9.35, 9.36

Q1 [Tue. 16/11, 16-17] Quiz 1 (based on lectures L1–L3 and exercises E1–E2)

L4 [Wed. 17/11, 10-12] Basic concepts in general relativity. Schwarzschild space-time.
Recommended reading: Guidry 7.1–7.2, 6.1–6.3, 9.1; Carroll 4.1, 4.7, 5.1–5.2; Cheng 6.1, 7.1; Schutz 7.1–7.3, 10.1; Wald 1.3–1.4, 4.1, 6.1

E3 [Wed. 17/11, 13-15] B&O Killing vector fields 2.63 [2.64], 2.65, 2.69 [2.70], Schwarzschild metric 2.72 [2.73], 2.73, Metrics, … 2.78 [2.81], 2.79 [2.82], 2.80 [2.83] (8 problems)

L5 [Thu. 18/11, 15-17] Schwarzschild space-time (continued).
Recommended reading: Guidry 9.3, 11.1–11.4; Carroll 5.3–5.4, 5.6–5.7, 6.1–6.3; Cheng 8.1–8.2, 14.1, 14.3, Schutz 10.2, 10.4–10.6, 11.2; Wald 6.2, 6.4, Chapter 9

L6 [Mon. 22/11, 13-15] Experimental tests of general relativity.
Recommended reading: Guidry 6.4–6.5, 9.2, 9.4–9.8; Carroll 5.5; Cheng 7.3.1, 8.3; Schutz 10.7, 11.1; Wald 6.3

E4 [Wed. 24/11, 10-12] B&O Frequency shifts 2.125 [2.132], 2.126 [2.133], Metrics, … 2.93 [2.94], 2.97 [2.98], 2.98 [2.100], Kruskal-Szekeres … 2.108 [2.109], Schwarzschild metric 2.76 [2.77] (7 problems)

Q2 [Wed. 24/11, 16-17] Quiz 2 (based on lectures L4–L6 and exercises E3–E4)

L7 [Thu. 25/11, 15-17] Einstein's field equations. The energy-momentum tensor.
Recommended reading: Guidry 8.5, 7.3; Carroll 4.2–4.6, 5.8; Cheng 14.2; Schutz 8.1–8.2, 10.3; Wald 4.3

L8 [Mon. 29/11, 13-15] Weak field limit.
Recommended reading: Guidry 8.1–8.3, 8.6–8.8, 22.2; Carroll 7.1–7.3; Cheng 6.2–6.3, 15.1–15.2; Schutz 8.3–8.4; Wald 4.4

E5 [Wed. 1/12, 10-12] B&O Maxwell's equations … 2.53 [2.57], 2.57 [2.61], 2.58 [2.62], 2.59 [2.106], Weak field … 2.114 [2.117], 2.116 [2.120], 2.118 [2.122] (7 problems)

L9 [Thu. 2/12, 15-17] Gravitational lensing. Gravitational waves.
Recommended reading: Guidry 17.7, 9.9, 22.1, 22.3–22.6; Carroll 8.6, 7.4–7.7; Cheng 7.2, 7.3.2, 15.3–15.4; Schutz Chapter 9; Wald 6.3, 4.4

E6 [Mon. 6/12, 13-15] B&O Gravitational lensing 2.119 [2.126], Metrics, … 2.82 [2.118], Gravitational waves 2.133, 2.135 (4 problems)

Q3 [Tue. 7/12, 16-17] Quiz 3 (based on lectures L7–L9 and exercises E5–E6)

L10 [Wed. 8/12, 10-12] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy.
Recommended reading: Guidry Chapters 16.1, 18; Carroll 8.1–8.3; Cheng 9.1, 9.3–9.4, 10.1; Schutz 12.1–12.2; Wald 5.1–5.2

E7 [Thu. 9/12, 15-17] B&O Metrics, … 2.81 [2.84], 2.103 [2.103], Frequency shifts 2.131 [2.138], Cosmology … 2.146 [2.147] (4 problems)
Additional problems: Guidry 19.1, 19.2, 19.6

L11 [Fri. 10/12, 10-12] Introductory cosmology (including the Friedmann–Lemaître–Robertson–Walker metric), including inflation and dark energy (continued).
Recommended reading: Guidry Chapter 16.2, 17.1–17.5, 19, 17.11–17.13, 21.3; Carroll 8.4–8.5, 8.7–8.8; Cheng 10.2–10.3, 14.4, 11.1–11.5; Schutz 12.3–12.4; Wald 5.3–5.4

E8 [Mon. 13/12, 13-15] B&O Cosmology … 2.147 [2.148], 2.144 [2.146], 2.148 [2.149], 2.149 [2.150] (4 problems)
Additional problems: Guidry 21.2, 21.5

L12 [Wed. 15/12, 10-12] Extra

E9 [Wed. 15/12, 13-15] Old exams

Q4 [Thu. 16/12, 16-17] Quiz 4 (based on lectures L10–L11 and exercises E7–E8)

FE [Mon. 17/1, 8-13] Final written exam

All lectures and all exercises will take place in room FD41 in Albanova.
All quizzes will take place on Canvas.
The final written exam will take place in rooms FA31 and FA32 in Albanova.

Problems in […] refer to the old numbering of problems in the student's manual Mattias Blennow and Tommy Ohlsson, *Relativity Theory – A Collection of 300 Problems in Special and General Relativity Theory with Complete Solutions*, KTH (2020).

Department of Physics

Final Written Exam in SH2372 General Relativity, 6 credits January 17, 2022, 08:00–13:00

Examiner: Prof. Tommy Ohlsson
(Telephone: 08-790 8261 • E-mail: tohlsson@kth.se)

Allowed aids: *Useful Formulas in General Relativity*

GOOD LUCK!

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1. A two-dimensional torus, denoted T^2, is a manifold that may be parametrized using two cyclic angular coordinates θ and φ. For two constants $R > \rho > 0$, we can define an embedding of T^2 into $\mathbb{R}^3$ as

$$x = (R + \rho \sin\varphi)\cos\theta, \qquad y = (R + \rho\sin\varphi)\sin\theta, \qquad z = \rho\cos\varphi.$$

Based on this embedding, compute the following:

a) The components g_{ab} of the induced metric tensor on T^2 resulting from the standard Euclidean metric in $\mathbb{R}^3$.

b) The Christoffel symbols of the Levi-Civita connection.

2. The two-dimensional de Sitter space dS_2 can be embedded in 1+2-dimensional Minkowski space and has the induced line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = r_0^2\left(d\tau^2 - \cosh^2\tau\, d\varphi^2\right),$$

where $r_0 > 0$ is a constant.

Which of the vector fields $A = \partial_\tau$, $B = \partial_\varphi$, and $C = -\cos\varphi\,\partial_\tau + \sin\varphi\tanh\tau\,\partial_\varphi$ are Killing vector fields on dS_2? Please note that you have to motivate your answer, it is simply not enough to just state that one vector field is a Killing vector field or not.

3. The spacetime outside of the Earth may be approximately described by the Schwarzschild spacetime with the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{r_*}{r}\right)dt^2 - \left(1 - \frac{r_*}{r}\right)^{-1}dr^2 - r^2 d\Omega^2,$$

where $g_{\mu\nu}$ is the Schwarzschild metric tensor, r_* is the Schwarzschild radius of the Earth (approximately 9 mm), and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. A GPS satellite is orbiting the Earth in free fall at a stationary radius $r = r_0$. The motion is assumed to occur in the plane $\theta = \pi/2$.

Hints: Christoffel symbols: $\Gamma^r_{tt} = \frac{r_*(r-r_*)}{2r^3}$, $\quad \Gamma^r_{t\varphi} = 0$, $\quad \Gamma^r_{\varphi\varphi} = -(r - r_*)\sin^2\theta$.
The relative speed v between two objects with 4-velocities U and V, respectively, has a γ factor of $\gamma = g(U, V)$.

a) Since r is constant, the motion will have a 4-velocity $U = \alpha\, \partial_t + \beta\, \partial_\varphi$. Find the values of the constants α and β.

b) Find an expression for the proper time it takes for the satellite to complete a full orbit around the Earth.

c) An observer is stationary at $r = r_0$ (note that this requires proper acceleration of this observer). At what speed will the satellite pass by the observer?

4. Gravitational waves can be described by perturbations $h_{\mu\nu}$ of the metric $g_{\mu\nu}$ such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $|h_{\mu\nu}| \ll 1$.

   Consider two particles at rest at $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (0, \ell, 0)$, respectively. A plus-polarized gravitational wave h_+ of frequency f and amplitude $h_0 \ll 1$ passes by, propagating in the z-direction, such that

$$(h_{\mu\nu}(t,x,y,z)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h_0 \sin\left[2\pi f\,(t - z)\right] \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Compute the distance d measured along the y-axis between the two particles (i.e., the spatial separation along the equal t hypersurface), as the wave passes, in terms of the initial distance ℓ.

   Hint: The geodesic equation for the displacement vector S_μ is given by $\frac{d^2 S_\mu}{dt^2} = \frac{1}{2}\frac{d^2 h_{\mu\nu}}{dt^2} S^\nu$.

5. Consider the Robertson–Walker line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a(t)^2 \left[dr^2 + S(r)^2 d\Omega^2\right],$$

   where $g_{\mu\nu}$ is the Robertson–Walker metric tensor, $a(t)$ is a scale factor that is some function of the universal time t, $S(r)$ is some function of a radial coordinate r, and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$.

   Hint: The following Christoffel symbols for the Robertson–Walker metric might be useful:

$$\Gamma^t_{\theta\theta} = a\dot{a}S^2, \quad \Gamma^r_{rt} = \Gamma^\theta_{\theta t} = \Gamma^\varphi_{\varphi t} = \dot{a}/a, \quad \Gamma^r_{\theta\theta} = -SS', \quad \Gamma^\theta_{\theta r} = \Gamma^\varphi_{\varphi r} = S'/S, \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta.$$

   a) Compute the $\theta\theta$ component of the Ricci tensor for the Robertson–Walker spacetime, i.e. $R_{\theta\theta}$.

   b) Using Einstein's field equations, derive an equation that relates the energy density ρ and pressure p for an ideal fluid to the functions $a(t)$ and $S(r)$.

6. One problem of the standard Big Bang model that the theory of inflation was introduced to cure is the so-called *flatness problem.*

   a) Illustrate this by deriving that the fractional deviation $\Delta\rho/\rho$ of the energy density ρ from the critical energy density ρ_c at any time in the evolution of the Universe is given by

$$\frac{\Delta\rho}{\rho} \equiv \frac{\rho - \rho_c}{\rho} = \frac{3\kappa}{8\pi G a^2 \rho},$$

   where κ is the curvature parameter.

   b) For a radiation-dominated universe, show that this implies that as one goes back in time, $\Delta\rho/\rho$ must decrease.

   c) Compute an estimate for $(\Delta\rho/\rho)_{t_{\mathrm{Pl}}}/(\Delta\rho/\rho)_{t_0}$, where $t_{\mathrm{Pl}} \simeq 5 \cdot 10^{-44}$ s is the Planck time and t_0 is the time today.

   Hint: The age of the Universe is about $4 \cdot 10^{17}$ s.

Useful Formulas in General Relativity

The *covariant derivatives* of a covariant vector A_ν and a contravariant vector A^ν are given by

$$A_{\nu;\mu} \equiv \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda, \qquad A^\nu_{;\mu} \equiv \nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda.$$

The *parallel transport equation* for a vector A^λ and the *geodesic equations* are given by

$$\dot{x}^\mu \nabla_\mu A^\lambda = \dot{A}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu A^\nu = 0, \qquad \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0.$$

The *torsion* T and the *curvature* R are defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \qquad R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z.$$

The components of the *torsion tensor* and the *Riemann curvature tensor* may be computed as

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}, \qquad R^\omega{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\omega_{\nu\lambda} - \partial_\nu \Gamma^\omega_{\mu\lambda} + \Gamma^\omega_{\mu\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\omega_{\nu\rho} \Gamma^\rho_{\mu\lambda}.$$

By definition, X is a Killing vector field if (for all indices μ and ν)

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0.$$

The *Lie derivative* of the metric tensor $g_{\mu\nu}$ with respect to a vector field $X = X^\lambda \partial_\lambda$ is given by

$$\mathcal{L}_X g_{\mu\nu} = X^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu} \partial_\mu X^\lambda + g_{\mu\lambda} \partial_\nu X^\lambda.$$

The particular value $r = r_* \equiv 2GM$ represents the *Schwarzschild radius*.
The *Kruskal–Szekeres metric* is given by

$$ds^2 = \frac{16\mu^2}{r} e^{(2\mu - r)/(2\mu)} du dv - r^2 d\Omega^2, \quad uv = (2\mu - r) e^{(r-2\mu)/(2\mu)} < \frac{2GM}{e}, \quad \mu \equiv GM.$$

The *Einstein–Hilbert action* is

$$\mathscr{S}_{\rm EH} = -\frac{M_{\rm Pl}^2}{2} \int R \sqrt{|g|}\, d^4x, \quad M_{\rm Pl} \equiv \frac{1}{\sqrt{8\pi G}}.$$

Einstein's field equations in matter are given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}.$$

The *energy-momentum tensor* is generally given by

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta \mathscr{S}_{\rm matter}}{\delta g^{\mu\nu}}.$$

For *an ideal (or perfect) fluid*, it holds that $T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu - p\, g_{\mu\nu}$.
Given two static observers A and B in a static spacetime, signals sent from A to B with frequencies f_A and f_B, respectively, will be redshifted according to

$$z = \frac{f_A}{f_B} - 1 = \frac{\varphi(x_B)}{\varphi(x_A)} - 1.$$

The two independent *Friedmann equations* are

$$\frac{\dot{a}(t)^2}{a(t)^2} = H(t)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a(t)^2}, \qquad \frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} (\rho + 3p)\,.$$

Solutions to Final Written Exam in General Relativity
SH2372 General Relativity • 6 ECTS credits
Monday January 17, 2022, 8:00 – 13:00

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1. a) The coordinate transformations are given by

$$x = (R + \rho \sin\varphi)\cos\theta, \quad y = (R + \rho \sin\varphi)\sin\theta, \quad z = \rho\cos\varphi.$$

Using these coordinate transformations, we find that

$$\begin{aligned} dx &= \rho\cos\varphi\cos\theta\, d\varphi - (R + \rho\sin\varphi)\sin\theta\, d\theta, \\ dy &= \rho\cos\varphi\sin\theta\, d\varphi + (R + \rho\sin\varphi)\cos\theta\, d\theta, \\ dz &= -\rho\sin\varphi\, d\varphi. \end{aligned}$$

Inserting into the line element gives

$$ds^2 = dx^2 + dy^2 + dz^2 = \rho^2\, d\varphi^2 + (R + \rho\sin\varphi)^2\, d\theta^2.$$

Thus, the components of the induced metric tensor on T^2 can be identified as

$$g_{\theta\theta} = (R + \rho\sin\varphi)^2, \quad g_{\varphi\varphi} = \rho^2, \quad g_{\theta\varphi} = g_{\varphi\theta} = 0,$$

which is what we wanted to compute in a).

b) A Lagrangian can be formulated as

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \rho^2\dot{\varphi}^2 + (R + \rho\sin\varphi)^2\dot{\theta}^2.$$

Using Euler–Lagrange equations for θ and φ, we find that

$$\begin{aligned} \ddot{\theta} + 2\frac{\rho\cos\varphi}{R + \rho\sin\varphi}\dot{\theta}\dot{\varphi} = 0 \quad &\Rightarrow \quad \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \frac{\rho\cos\varphi}{R + \rho\sin\varphi}, \\ \ddot{\varphi} - \frac{(R + \rho\sin\varphi)\cos\varphi}{\rho}\dot{\theta}^2 = 0 \quad &\Rightarrow \quad \Gamma^\varphi_{\theta\theta} = -\frac{(R + \rho\sin\varphi)\cos\varphi}{\rho}, \end{aligned}$$

which are the only non-zero Christoffel symbols and what we wanted to compute in b).

2. The line element is given by

$$ds^2 = r_0^2\left(d\tau^2 - \cosh^2\tau\, d\varphi^2\right),$$

where $r_0 > 0$ is a constant. The components of the induced metric tensor can be identified from this line element as

$$g_{\tau\tau} = r_0^2, \quad g_{\varphi\varphi} = -r_0^2\cosh^2\tau, \quad g_{\tau\varphi} = g_{\varphi\tau} = 0.$$

Note that these metric components do not depend on the coordinate φ. This means that the coordinate vector field $B = \partial_\varphi$ is a Killing vector field.

In general, to check if a vector field K is a Killing vector field or not can be performed explicitly by computing the Lie derivative of the metric tensor with respect to the vector field K under investigation:

$$\mathcal{L}_K g_{\mu\nu} = K^\lambda\partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu K^\lambda + g_{\mu\lambda}\partial_\nu K^\lambda.$$

Explicitly, using that the metric components are independent of φ and symmetric, we have

$$\begin{aligned} \mathcal{L}_K g_{\tau\tau} &= K^\lambda\partial_\lambda g_{\tau\tau} + g_{\lambda\tau}\partial_\tau K^\lambda + g_{\tau\lambda}\partial_\tau K^\lambda = 2g_{\tau\tau}\partial_\tau K^\tau, \\ \mathcal{L}_K g_{\varphi\varphi} &= K^\lambda\partial_\lambda g_{\varphi\varphi} + g_{\lambda\varphi}\partial_\varphi K^\lambda + g_{\varphi\lambda}\partial_\varphi K^\lambda = K^\tau\partial_\tau g_{\varphi\varphi} + 2g_{\varphi\varphi}\partial_\varphi K^\varphi, \\ \mathcal{L}_K g_{\tau\varphi} &= K^\lambda\partial_\lambda g_{\tau\varphi} + g_{\tau\lambda}\partial_\varphi K^\lambda + g_{\lambda\varphi}\partial_\tau K^\lambda = g_{\tau\tau}\partial_\varphi K^\tau + g_{\varphi\varphi}\partial_\tau K^\varphi. \end{aligned}$$

First, for $A = \partial_\tau \Leftrightarrow A^\tau = 1 \wedge A^\varphi = 0$, we find that

$$\mathcal{L}_A g_{\tau\tau} = 2g_{\tau\tau}\partial_\tau A^\tau = 0, \quad \mathcal{L}_A g_{\varphi\varphi} = A^\tau \partial_\tau g_{\varphi\varphi} \neq 0, \quad \mathcal{L}_A g_{\tau\varphi} = g_{\tau\tau}\partial_\varphi A^\tau = 0,$$

which means that A is not a Killing vector field. Second, for $B = \partial_\varphi \Leftrightarrow B^\tau = 0 \wedge B^\varphi = 1$, we find that

$$\mathcal{L}_B g_{\tau\tau} = 0, \quad \mathcal{L}_B g_{\varphi\varphi} = 2g_{\varphi\varphi}\partial_\varphi B^\varphi = 0, \quad \mathcal{L}_B g_{\tau\varphi} = g_{\varphi\varphi}\partial_\tau B^\varphi = 0,$$

which means that B is a Killing vector field. Third, for $C = -\cos\varphi\,\partial_\tau + \sin\varphi\tanh\tau\,\partial_\varphi \Leftrightarrow C^\tau = -\cos\varphi \wedge C^\varphi = \sin\varphi\tanh\tau$, we find that

$$\begin{aligned}
\mathcal{L}_C g_{\tau\tau} &= 2g_{\tau\tau}\partial_\tau C^\tau = 0,\\
\mathcal{L}_C g_{\varphi\varphi} &= C^\tau\partial_\tau g_{\varphi\varphi} + 2g_{\varphi\varphi}\partial_\varphi C^\varphi = -\cos\varphi\,\partial_\tau(-r_0^2\cosh^2\tau) + 2(-r_0^2\cosh^2\tau)\partial_\varphi(\sin\varphi\tanh\tau)\\
&= r_0^2\cos\varphi\cdot 2\cosh\tau\cdot\sinh\tau - 2r_0^2\cosh^2\tau\tanh\tau\cos\varphi = 0,\\
\mathcal{L}_C g_{\tau\varphi} &= g_{\tau\tau}\partial_\varphi C^\tau + g_{\varphi\varphi}\partial_\tau C^\varphi = r_0^2\partial_\varphi(-\cos\varphi) + (-r_0^2\cosh^2\tau)\partial_\tau(\sin\varphi\tanh\tau)\\
&= r_0^2\sin\varphi - r_0^2\cosh^2\tau\sin\varphi\cdot\frac{1}{\cosh^2\tau} = 0,
\end{aligned}$$

which means that C is a Killing vector field. In conclusion, the vector fields B and C are Killing vector fields, but the vector field A is not a Killing vector field.

3. a) The worldline of the GPS satellite will be given by

$$t = \alpha s, \quad r = r_0, \quad \varphi = \beta s,$$

since the 4-velocity is $U = \alpha\,\partial_t + \beta\,\partial_\varphi$, where α and β are constants. It follows that the 4-acceleration is given by

$$A = \nabla_U U = (\alpha\nabla_t + \beta\nabla_\varphi)(\alpha\,\partial_t + \beta\,\partial_\varphi) = \left(\alpha^2\Gamma^\lambda_{tt} + 2\alpha\beta\Gamma^\lambda_{t\varphi} + \beta^2\Gamma^\lambda_{\varphi\varphi}\right)\partial_\lambda = 0$$

for the satellite to be freely falling. Given the Christoffel symbols

$$\Gamma^r_{tt} = \frac{r_*(r-r_*)}{2r^3}, \quad \Gamma^r_{t\varphi} = 0, \quad \Gamma^r_{\varphi\varphi} = -(r-r_*)\sin^2\theta$$

of the Schwarzschild metric, the only non-trivial component of the 4-acceleration is the r component (for $r = r_0$ and $\theta = \pi/2$) such that

$$\alpha^2\Gamma^r_{tt} + 2\alpha\beta\Gamma^r_{t\varphi} + \beta^2\Gamma^r_{\varphi\varphi} = \alpha^2\frac{r_*(r_0-r_*)}{2r_0^3} - \beta^2(r_0-r_*) = 0,$$

which leads to

$$\beta = \alpha\sqrt{\frac{r_*}{2r_0^3}}.$$

Also, requiring the normalization condition $1 = g(U,U)$ yields

$$1 = g(U,U) = g_{tt}\alpha^2 + g_{\varphi\varphi}\beta^2 = \left(1 - \frac{r_*}{r_0}\right)\alpha^2 - r_0^2\beta^2.$$

Inserting $\beta = \alpha\sqrt{\frac{r_*}{2r_0^3}}$ into this condition, we find that

$$1 = \left(1 - \frac{r_*}{r_0}\right)\alpha^2 - r_0^2\cdot\alpha^2\frac{r_*}{2r_0^3} = \alpha^2\left(1 - \frac{3r_*}{2r_0}\right),$$

which means that

$$\alpha = \frac{1}{\sqrt{1 - \frac{3r_*}{2r_0}}}, \quad \beta = \alpha\sqrt{\frac{r_*}{2r_0^3}} = \frac{1}{r_0}\sqrt{\frac{r_*}{2r_0 - 3r_*}}.$$

Thus, we obtain

$$\alpha = \left(1 - \frac{3r_*}{2r_0}\right)^{-1/2}, \quad \beta = \frac{1}{r_0}\left(\frac{r_*}{2r_0 - 3r_*}\right)^{1/2},$$

which is what we wanted to find in a).

b) For the satellite to complete a full orbit, φ needs to change by 2π. The proper time $\Delta\tau$ that it takes for the satellite to travel an angle $\Delta\varphi$ is given by

$$\frac{\Delta\varphi}{\Delta\tau} = \dot{\varphi},$$

which implies that

$$\Delta\tau = \frac{\Delta\varphi}{\dot{\varphi}} = \{\dot{\varphi} = \beta\} = \frac{\Delta\varphi}{\beta}.$$

Therefore, using $\Delta\varphi = 2\pi$ and $\beta = \frac{1}{r_0}\left(\frac{r_*}{2r_0 - 3r_*}\right)^{1/2}$, we find that

$$\Delta\tau = 2\pi r_0 \sqrt{\frac{2r_0 - 3r_*}{r_*}},$$

which is what we wanted to find in b).

c) The stationary observer has the 4-velocity $V = \alpha_0\, \partial_t$. Using the normalization condition for this 4-velocity, we have

$$1 = g(V, V) = g_{tt}\alpha_0^2 = \left(1 - \frac{r_*}{r_0}\right)\alpha_0^2,$$

which implies that

$$\alpha_0 = \sqrt{\frac{r_0}{r_0 - r_*}}.$$

Now, using the hint, the γ factor is given by

$$\gamma = g(U, V) = g_{tt}\alpha\alpha_0 = \left(1 - \frac{r_*}{r_0}\right) \cdot \frac{1}{\sqrt{1 - \frac{3r_*}{2r_0}}} \cdot \sqrt{\frac{r_0}{r_0 - r_*}} = \sqrt{\frac{2(r_0 - r_*)}{2r_0 - 3r_*}}.$$

Since $\gamma = \frac{1}{\sqrt{1 - v^2}}$, we obtain

$$v = \sqrt{\frac{r_*}{2(r_0 - r_*)}},$$

which is the speed that we wanted to compute in c).

4. Let us consider two particles, which are influenced by a plus-polarized gravitational wave along the z-direction such that

$$(h_{\mu\nu}) = \mathrm{diag}(0, h_+, -h_+, 0), \quad h_+ = h_0 \sin\left[2\pi f(t - z)\right], \quad h_\times = 0,$$

where $h_0 \ll 1$ is the amplitude and f is the frequency.

Then, in fact, the geodesic equation for the displacement vector S_μ is given by

$$\frac{d^2 S_\mu}{dt^2} = \frac{1}{2}\frac{d^2 h_{\mu\nu}}{dt^2} S^\nu,$$

where $h_{\mu\nu}$ is the given gravitational wave. We observe that this gravitational wave affects neither S_0 nor S_3. Thus, the only effect on the geodesics is taking place in the x- and y-directions. Without loss of generality, we can therefore assume that $z = 0$. In this case, the gravitational wave is plus-polarized only, i.e., $h_\times = 0$, so that the geodesic equation simplifies to two equations for $S_1 = -S^1$ and $S_2 = -S^2$, namely

$$\frac{d^2 S_1}{dt^2} = \frac{(2\pi f)^2}{2} h_0 \sin(2\pi f t)\, S_1, \qquad \frac{d^2 S_2}{dt^2} = -\frac{(2\pi f)^2}{2} h_0 \sin(2\pi f t)\, S_2,$$

which can be solved perturbatively in h_0. Up to first-order in h_0, we obtain

$$S_1(t) = S_1(0)\left[1 - \frac{1}{2} h_0 \sin(2\pi f t) + \ldots\right], \qquad S_2(t) = S_2(0)\left[1 + \frac{1}{2} h_0 \sin(2\pi f t) + \ldots\right].$$

Next, the measured distance $\Delta y \equiv S_2(t)$ between the two particles, which was initially the distance $\Delta y_0 \equiv S_2(0)$ along the y-direction, will be

$$\frac{\Delta y}{\Delta y_0} = \frac{S_2(t)}{S_2(0)} \simeq 1 + \frac{1}{2} h_0 \sin(2\pi f t),$$

which means that the relative distance $\delta y \equiv \Delta y - \Delta y_0$ between the two particles oscillate with f. This does not mean that the positions of the particle coordinates change, but the coordinates themselves oscillate.

Finally, assuming d to be the measured distance Δy and ℓ the initial distance Δy_0 between the two particles, we obtain

$$\frac{d}{\ell} = 1 + \frac{1}{2} h_0 \sin(2\pi f t) \quad \Rightarrow \quad d = \left[1 + \frac{1}{2} h_0 \sin(2\pi f t)\right] \ell,$$

which is what we wanted to compute.

5. a) The $\theta\theta$ component of the Ricci tensor is given by

$$\begin{aligned}
R_{\theta\theta} &= R^\mu{}_{\theta\mu\theta} = \partial_\mu \Gamma^\mu_{\theta\theta} - \partial_\theta \Gamma^\mu_{\mu\theta} + \Gamma^\lambda_{\theta\theta}\Gamma^\mu_{\mu\lambda} - \Gamma^\lambda_{\mu\theta}\Gamma^\mu_{\theta\lambda} \\
&= \partial_t \Gamma^t_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\varphi_{\varphi\theta} + \Gamma^t_{\theta\theta}\Gamma^\mu_{\mu t} + \Gamma^r_{\theta\theta}\Gamma^\mu_{\mu r} - \Gamma^t_{\theta\theta}\Gamma^\theta_{\theta t} - \Gamma^\theta_{t\theta}\Gamma^t_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} - \Gamma^\theta_{r\theta}\Gamma^r_{\theta\theta} - \Gamma^\varphi_{\varphi\theta}\Gamma^\varphi_{\theta\varphi} \\
&= \partial_t \Gamma^t_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\varphi_{\varphi\theta} + \Gamma^t_{\theta\theta}\left(\Gamma^r_{rt} + \Gamma^\theta_{\theta t} + \Gamma^\varphi_{\varphi t}\right) + \Gamma^r_{\theta\theta}\left(\Gamma^\theta_{\theta r} + \Gamma^\varphi_{\varphi r}\right) \\
&- 2\Gamma^t_{\theta\theta}\Gamma^\theta_{\theta t} - 2\Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} - \left(\Gamma^\varphi_{\varphi\theta}\right)^2.
\end{aligned}$$

Using the Christoffel symbols

$$\begin{aligned}
&\Gamma^t_{\theta\theta} = a\dot{a}S^2, \quad \Gamma^r_{rt} = \Gamma^\theta_{\theta t} = \Gamma^\varphi_{\varphi t} = \dot{a}/a, \quad \Gamma^r_{\theta\theta} = -SS', \quad \Gamma^\theta_{\theta r} = \Gamma^\varphi_{\varphi r} = S'/S, \\
&\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta,
\end{aligned}$$

we find that

$$\begin{aligned}
\partial_t \Gamma^t_{\theta\theta} &= \partial_t(a\dot{a}S^2) = (\dot{a}\dot{a} + a\ddot{a})S^2 = (\dot{a}^2 + a\ddot{a})S^2, \\
\partial_r \Gamma^r_{\theta\theta} &= \partial_r(-SS') = -(S'S' + SS'') = -(S'^2 + SS''), \\
\partial_\theta \Gamma^\varphi_{\varphi\theta} &= \partial_\theta \cot\theta = -\frac{1}{\sin^2\theta}.
\end{aligned}$$

Inserting these intermediate results and further Christoffel symbols into $R_{\theta\theta}$, we obtain

$$\begin{aligned}
R_{\theta\theta} &= (\dot{a}^2 + a\ddot{a})S^2 + (-1)(S'^2 + SS'') - \left(-\frac{1}{\sin^2\theta}\right) + a\dot{a}S^2 \cdot 3\frac{\dot{a}}{a} + (-SS')\cdot 2\frac{S'}{S} \\
&- 2a\dot{a}S^2 \cdot \frac{\dot{a}}{a} - 2(-SS')\frac{S'}{S} - \cot^2\theta = (2\dot{a}^2 + a\ddot{a})S^2 - (S'^2 + SS'') + 1.
\end{aligned}$$

b) Einstein's field equations can be written as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu} \quad \Leftrightarrow \quad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} T\, g_{\mu\nu}\right),$$

where $T \equiv T^\mu_\mu = \rho - 3p$ for an ideal fluid. Thus, the $\theta\theta$ component yields

$$R_{\theta\theta} = 8\pi G \left(T_{\theta\theta} - \frac{1}{2} T\, g_{\theta\theta} \right) = 4\pi G\, g_{\theta\theta}(p - \rho) = \{g_{\theta\theta} = -a^2 S^2\} = -4\pi G\,(p - \rho) a^2 S^2.$$

Now, identifying the two results for $R_{\theta\theta}$, we find that

$$(2\dot{a}^2 + a\ddot{a}) S^2 - (S'^2 + SS'') + 1 = -4\pi G\,(p - \rho) a^2 S^2,$$

which leads to

$$2\dot{a}^2 + a\ddot{a} - \frac{1}{S^2}(SS'' + S'^2 - 1) = 4\pi G(\rho - p) a^2.$$

In fact, the expression $\frac{1}{S^2}(SS'' + S'^2 - 1)$ can be written as $2S''/S$, and therefore, we finally obtain the equation

$$2\dot{a}(t)^2 + a(t)\ddot{a}(t) - 2\frac{S''(r)}{S(r)} = 4\pi G(\rho - p) a(t)^2,$$

which is what we wanted to derive.

6. a) The Friedmann equation for H is given by

$$\frac{\dot{a}}{a} = H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}$$

with the critical energy density defined as

$$\rho_c \equiv \frac{3H^2}{8\pi G} \quad \Leftrightarrow \quad H^2 = \frac{8\pi G}{3}\rho_c.$$

Using these two equations, we find that

$$\frac{8\pi G}{3}\rho_c = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2},$$

which means that

$$\rho - \rho_c = \frac{3\kappa}{8\pi G a^2}.$$

Thus, we obtain

$$\frac{\Delta\rho}{\rho} \equiv \frac{\rho - \rho_c}{\rho} = \frac{3\kappa}{8\pi G a^2 \rho},$$

which is what we wanted to derive.

b) For a radiation-dominated universe, we have $\rho \propto a^{-4} = \{a(t) \propto \sqrt{t}\} = t^{-2}$, so this means that $\Delta\rho/\rho \propto (\sqrt{t})^{-2}(t^{-2})^{-1} = t^{-1}t^2 = t$, which says that deviation from flatness grows smaller as time is extrapolated backwards, which is what we wanted to show.

c) Finally, using $t_0 \simeq 4 \cdot 10^{17}$ s and $t_{\rm Pl} \simeq 5 \cdot 10^{-44}$ s, we obtain

$$(\Delta\rho/\rho)_{t_{\rm Pl}}/(\Delta\rho/\rho)_{t_0} \simeq \frac{5 \cdot 10^{-44}\ \text{s}}{4 \cdot 10^{17}\ \text{s}} \sim 10^{-61},$$

which is an incredibly small number and what we wanted to estimate.

Resultat av: General Relativity, SH2372, ht 2021

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Publicerad under: 2022-01-16 - 2022-01-28
Antal svar: 14
Procent av kursdeltagarna som svarat: 53%
Kontaktperson: Tommy Ohlsson

What is your overall impression of the course?

14 svarande

Very positive	3	21%
Quite postive	10	71%
Neutral - no opinion	1	7%
Quite negative	0	0%
Very negative	0	0%

- *Really interesting course, however a bit fast for me maybe. (Quite postive)*
- *Very interesting content and it was generally very clear what we were expected to learn. (Quite postive)*

How would you rate the difficulty of the course?

14 svarande

Very difficult	1	7%
Quite difficult	13	92%
Average	0	0%
Easy	0	0%
Very easy	0	0%

- *There are lots of notions which were completely new for me, in quite a short time. The first courses about all the mathematical notations etc*

were too fast for me as I was discovering these here. (Quite difficult)
- In the beginning of the course, especially during the differential geometry part, there was a lot of material to cover - much more per lecture than usual which made it difficult. As we approached later parts of the course it felt like the pace slowed to being more normal. That is not to say that the pace became slow or the material easy, rather the pace was steady with difficult but doable material (Quite difficult)
- Difficult, but in a good way. After all, GR is not an easy subject. (Quite difficult)

Has there been much overlap with other courses?

14 svarande

Far too much overlap	0	0%
Some overlap, but it was useful to go over the topics again	11	78%
Mostly unnecessary overlap	1	7%
No overlap	2	14%

- As the same notations were used in special relativity, it helped a lot to get familiar with these to have two approaches. (Some overlap, but it was useful to go over the topics again)
- The first lecture contained concepts introduced in other courses, but this was useful for their generalization in following lectures. (Some overlap, but it was useful to go over the topics again)
- There was an overlap with Research Methodology in Physics which had mandatory seminars. (Some overlap, but it was useful to go over the topics again)
- Special relativity (obviously) (Some overlap, but it was useful to go over the topics again)

How were the quizzes?

14 svarande

Very difficult	1	7%
Difficult	11	78%
Average	2	14%
Easy	0	0%
Very easy	0	0%

- Some questions were really specific, on very particular aspects of the course. Since the quizzes were right after the lectures, I had little perspective on the course and get confused quite a lot. (Difficult)
- The problem with the quizzes was that the difficulty was more based on making sure to read the question correctly, rather than developing a deeper understanding of the material. In my opinion this is better achieved with weekly hand in problems similar to those on the exam,

since they really prepare you for tackling such problems. (Difficult)
- Many questions felt a lot like "trick questions" and/or misleading and could've been interpreted differently from what the teacher wanted, and therefore felt very unfair. For example, using words like "form" (of an equation for example), which has ambiguous meaning without further explanation. Another example is using the word "quadrant" in the context of whether it's true or not that in Kruskal-Szekeres coordinates, white hole is in 2 "quadrants". This is misleading, since quadrant is used frequently as meaning the quadrants of the coordinates, and in T-X coordinates, the white hole does in fact lie in two quadrants (3rd and 4th). A better word would've been "regions", which would cause no confusion. While I like the idea of continuous examination, the quiz-format like this was in my opinion not very good. An alternative could be weekly quiz but with written answers, so you don't have to get zero points because the question was unclear and was interpreted differently. (Difficult)
- It varied a lot, there were easier ones (which i would say was avarage) and then there were difficult ones (which was very hard) (Difficult)
- It felt like some questions were unrelated to general relativity, and so it was quite frustrating that answering incorrectly on these could affect the grade. (Difficult)
- In a good way. (Difficult)
- You had to stay focused all the time even tho you don't have time sometimes. (Difficult)

How was the final written exam?

14 svarande

Very difficult	1	7%
Difficult	8	57%
Average	5	35%
Easy	0	0%
Very easy	0	0%

- In my opinion, I believe that some problems went a bit far on the "specific things that might not have been properly discussed on the course" (Very difficult)
- This exam was very fair and fun, and tested us on relevant material. (Difficult)
- There broad range of questions made it difficult. The questions themselves weren't too hard. (Difficult)
- It was the right level of difficulty for me. Actually it was not surprising in a bad way, it was on what we have done so far (course and exercise sessions). (Average)
- I was very unsure about how the difficulty of the exam would turn out, only having the problem book as reference. However, I think that it turned out good! (Average)
- I think it was a fair exam (Average)

What is your opinion about the course description and the administration of the course?

14 svarande

Very good	3	21%
Good	9	64%
Average	2	14%
Poor	0	0%
Very poor	0	0%

- I do not really focus on this and have no strong opinion here (Average)

What is your opinion about the course literature?

14 svarande

Very good	2	14%
Good	6	42%
Average	6	42%
Poor	0	0%
Very poor	0	0%

- Guidry's book was book, but sometimes I think that on certain topics, it is a bit shallow. I also enjoyed having B&O book for support in doing exercises. (Good)
- I preferred the book by Carroll to the one by Guidry (Average)

How were the exercises? (Linda Tenhu)

13 svarande

Very good	6	46%
Good	3	23%
Average	4	30%
Poor	0	0%
Very poor	0	0%

- I did not attend (?)
- The exercises were very good and I appreciated how Linda referred certain equations to the book, and her methodical explanation of the problem solutions. (Very good)
- An amazing TA, she was always clear and open to receive questions. (Very good)
- I had almost all the time another course in the meantime so I worked quite a lot on my own, so I am not sure to have an informed judgment. (Good)

- I liked Linda's commitment and the problems that were solved were solved thoroughly. However, especially in the beginning, I think that the pace might have been a bit too slow. I know that it is important to make sure that everyone follows the solution, but I think it became a very stark contrast in pace and difficulty between the lectures and exercises. (Average)

How were the lectures? (Tommy Ohlsson)

14 svarande

Very good	5	35%
Good	5	35%
Average	2	14%
Poor	2	14%
Very poor	0	0%

- I enjoyed all of the lectures. I sometimes had the feeling of doing a "speedrun" of the courses, but this is something that is to be expected considering the material to see in the amount of time. (Very good)
- Blackboard writing > pdf slides on projector (Good)
- Same here, but the few ones I attended were good. The notes on Canvas truly saved me for this course. (Good)
- Very nice that Tommy learned and referred to people going to the lectures by name. To me this has not happened before. (Good)
- First part of the lectures were very rushed. The remaining lectures felt better. (Average)
- The setup with the lecture notes on the projector was not satisfactory. Many times it was difficult to follow, and so I skipped going to the lectures and studied by myself. This subject is however more interesting to learn in conjunction with discussions, and so I would have preferred to have more dynamic and well thought out lectures. (Poor)

Please enter any further comments on the course below.

- I find the grading system quite tough compared to some other classes. For me, the quizzes are a good idea but make them represent almost half of the final grade seems to be far too much. In other courses, this would have been only bonus points. Maybe add some homework problems (similar to the ones solved in exercise sessions) in the continuous grading ? Last comment, offering the 300 problems book is really nice, thanks !
- In my opinion the concept of quizzes where the points are final and cannot be replaced on the exam causes unnecessary stress and exchanges the learning opportunity that e.g. hand in tasks can be to more tests of being able to check the lecture notes and read the questions correctly.
- The course credit of 6.0hp does not match the time required to put into the course. It definitely feels more like a 7.5hp course.
- The lecture notes could have been better. They were simply copied off

of Mattias Blennow's lecture videos, but sometimes left out important details. This was frustrating. But other than that, this was a very fun course and I am very happy with what I have learned!
- Maybe a fifth quiz for those who miss a quiz. All other then have the chance to improve, such that you only take the best 4 out of 5 quizzes you offer.

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